Research Article

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Boundary layer analysis of nonlinear reaction-diffusion equations in a smooth domain

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Abstract: In this article, we consider a singularly perturbed nonlinear reaction-diffusion equation whose solutions display thin boundary layers near the boundary of the domain. We fully analyse the singular behaviours of the solutions at any given order with respect to the small parameter ε, with suitable asymptotic expansions consisting of the outer solutions and of the boundary layer correctors. The systematic treatment of the nonlinear reaction terms at any given order is novel along the singular perturbation analysis. We believe that the analysis can be suitably extended to other nonlinear problems.

Keywords: Boundary layers, singular perturbations, reaction-diffusion, boundary fitted coordinates

MSC 2010: 35B25, 35C20, 76D10, 35K05

1 Introduction

Nonlinear reaction-diffusion equations arise in many areas in systems consisting of interacting components. The equations describe, e.g., chemical reactions, pattern-formation, population dynamics, predator-prey equations, and competition dynamics in biological systems (see, e.g., [7, 11, 12, 31–33, 39]). One can consider a typical form of systems of reaction-diffusion equations in the form

$$\mathbf{u}_t = D\Delta\mathbf{u} + \mathbf{g}(\mathbf{u}), \tag{1.1}$$

where $\mathbf{g} = \mathbf{g}(\mathbf{u})$ describes a change or a local reaction of the state $\mathbf{u}$ and D represents a diffusion coefficient matrix. It is also possible that the reaction $\mathbf{g}$ may depend on the spatial domain variable $\mathbf{x}$ and of a derivative of $\mathbf{u}$, i.e., $\mathbf{g} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u})$.

In real applications like a fast reaction system, the magnitude of some coefficients in the diffusion matrix D is relatively small and hence the system can be singularly perturbed.

In this article, for the singular perturbation and boundary layer analysis aimed here, we consider the steady state system of (1.1) and study the following scalar nonlinear singularly perturbed problem which can serve as a guide for more general systems:

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + g(u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{at } \partial\Omega. \end{cases} \tag{1.2}$$

Here, $0 < \varepsilon \ll 1$, Ω is a general smooth domain, $f = f(x, y)$ and $g = g(u)$ are given smooth functions with

$$g(0) = 0, \qquad g'(u) \geq \lambda > 0 \quad \text{for all } u \in \mathbb{R}. \tag{1.3}$$

For example, $g(u) = u^3 + \lambda u$.

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For small $\varepsilon > 0$, the solutions to (1.2) display thin sharp transition layers called boundary layers which are formed due to the discrepancies between the limit solutions when $\varepsilon = 0$ (see (2.3) below) and the boundary conditions in (1.2). The discrepancies are inevitable because the limit problem (see (2.2) below) loses high order derivatives and hence in general its solutions cannot meet the boundary conditions. Then, the small diffusion term $-\varepsilon\Delta u^\varepsilon$ smoothes out the discrepancies, which leads to sharp transition boundary layers.

Another motivation of studying boundary layers is the vanishing viscosity problem in fluid dynamics, see, e.g., [2, 5, 6, 8, 13, 23–25, 27, 28, 30, 35–37]. The typical question is on the behaviour of the Navier–Stokes flows at small viscosity, i.e., the limit behaviour or convergence to Euler flows as the viscosity tends to zero. The boundary layers play a crucial role for connecting the Navier–Stokes and Euler flows and they also do so for the singular perturbation analysis in the nonlinear reaction-diffusion equations considered here.

An additional motivation comes from the computational aspects in numerical simulations. Due to the thin boundary layers, the computational meshes are classically refined near the boundary $\partial\Omega$ and this causes high cost in the simulations. Rather than refining meshes we propose to enrich with suitable boundary layer correctors the Galerkin or finite element space (or finite volume space). Then, we are able to use a coarse mesh and this reduces substantially the computational cost. See, e.g., [17, 18, 20–22, 38, 41] for the method of spaces enriched with boundary layer correctors. For singular perturbations analysis, see [15, 26, 42, 44] and also the recent review article [14]. See other perspectives in singular perturbations and boundary layers in [3, 4, 9, 10, 16, 19, 29, 43].

In what follows, we discuss the problems posed on a channel domain in Section 2 which is relatively easier to handle thanks to the simple geometry of the boundary. In Section 3, we cast the nonlinear reaction-diffusion equations in a general domain. We need to take into account the geometrical properties, like curvature, using the boundary fitted coordinates. Throughout this paper, we systematically handle the nonlinear term g along the singular perturbation analysis at any orders. This nonlinear treatment can apply to other nonlinear problems.

For the analysis below, we shall consider the Sobolev spaces $H^s(\Omega)$ and we define the weighted energy norm,

$$\|u\|_\varepsilon = (\varepsilon\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

An exponentially small term, denoted *e.s.t.*, is a function whose norm in all Sobolev spaces $H^s(\Omega)$ is exponentially small with, for each s, a bound of the form $c_1 e^{-c_2/\varepsilon^\gamma}$, $c_1, c_2, \gamma > 0$, with c_i, γ depending possibly on s.

2 Channel domains

For general domains, which will be studied in Section 3, we consider the domains with smooth boundaries. Since the boundary layer correctors act locally in the inward direction normal to the boundaries, transforming the Cartesian coordinate into the so-called boundary fitted one, the boundary layers can be described in channel domains, which are relatively easy to analyse. We thus consider first the simpler case of channel domains, which possess boundary layers only on one side at a flat boundary.

Let us consider the problem in a channel domain as follows:

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + g(u^\varepsilon) = f & \text{in } \Omega = (0, L_1) \times (0, L_2), \\ u^\varepsilon = 0 & \text{at } x = 0, L_1, \\ u^\varepsilon(x, y) = u^\varepsilon(x, y + L_2) & \text{in } \Omega_\infty = (0, L_1) \times \mathbb{R}, \end{cases} \tag{2.1}$$

where $f = f(x, y)$ is smooth and L_2- periodic in y. Then, the limit problem reads

$$g(u^0) = f \quad \text{in } \Omega. \tag{2.2}$$

Since g is invertible, we write

$$u^0 = g^{-1}(f). \tag{2.3}$$

To give an idea on how to construct the boundary layers, for now we assume

$$f = 0 \quad \text{at } x = L_1, \tag{2.4}$$

which, as we will see, reduces the boundary layer at $x = L_1$, so that only the boundary layer at $x = 0$ persists.

Thanks to (1.3) and (2.3), $0 \le \lambda(u^0)^2 \le (g(u^0) - g(0))u^0 = fu^0$, and hence

$$u^0 = 0 \quad \text{at } x = L_1 \quad \text{and} \quad u^0(x, y) = u^0(x, y + L_2). \tag{2.5}$$

2.1 Boundary layer analysis at order ε^0

We now construct a zeroth order corrector to account for the discrepancy between u^ε and u^0 at $x = 0$. Formally, substituting $u^\varepsilon \sim u^0 + \theta^0$ in (2.1) and subtracting (2.2) from (2.1), we find that

$$-\varepsilon\Delta(u^0 + \theta^0) + g(u^0 + \theta^0) - g(u^0) = 0.$$

Using the stretched variable $\bar{x} = x/\sqrt{\varepsilon}$ and dropping non-stiff small terms, we find the zeroth order corrector equation for θ^0:

$$-\varepsilon\theta^0_{xx} + g(u^0 + \theta^0) - g(u^0) = 0.$$

However, in general $u^\varepsilon - u^0$ does not satisfy the boundary condition in (2.1), and hence at the boundary $x = 0$, so we propose a boundary layer corrector θ^0 satisfying

$$\begin{cases} -\varepsilon\theta^0_{xx} + g(u^0 + \theta^0) - g(u^0) = 0 & \text{in } \Omega, \\ \theta^0 = -u^0(0, y) & \text{at } x = 0, \\ \theta^0 = 0 & \text{at } x = L_1. \end{cases} \tag{2.6}$$

Although θ^0 is not known explicitly, unlike in many linear problems, we can derive pointwise estimates for θ^0.

Lemma 2.1. *The corrector θ^0 satisfies*

$$|\theta^0(x, y)| \le |u^0(0, y)| \exp\Big(-\sqrt{\frac{\lambda}{\varepsilon}}x\Big). \tag{2.7}$$

Proof. Setting $\bar{\theta}^0 = |u^0(0, y)| \exp(-\sqrt{\lambda}x/\sqrt{\varepsilon})$, writing $\tilde{\theta}^0 = \theta^0 - \bar{\theta}^0$ and then substituting in (2.6), we obtain $-\varepsilon\tilde{\theta}^0_{xx} + g(u^0 + \theta^0) - g(u^0) - \lambda\bar{\theta}^0 = 0$. Since $g'(\eta) - \lambda \ge 0$ for all $\eta \in \mathbb{R}$ and thanks to the mean value theorem, we find, for some η_1 with $|\eta_1 - u^0| < |\theta^0|$, that

$$-\varepsilon\tilde{\theta}^0_{xx} + g'(\eta_1)\tilde{\theta}^0 = (-g'(\eta_1) + \lambda)\bar{\theta}^0 \le 0. \tag{2.8}$$

Multiplying (2.8) by $\tilde{\theta}^0_+ = \max\{\tilde{\theta}^0, 0\}$, integrating over $(0, L_1)$ and noting that $\tilde{\theta}^0_+ = 0$ at $x = 0, L_1$, we obtain

$$\varepsilon \int_0^{L_1} ((\tilde{\theta}^0_+)_x)^2 + \lambda \int_0^{L_1} (\tilde{\theta}^0_+)^2 \le 0.$$

This implies $\tilde{\theta}^0_+ = 0$ and thus $\theta^0 - \bar{\theta}^0 = \tilde{\theta}^0 \le 0$. On the other hand, considering this time $\tilde{\theta}^0 = -\theta^0 - \bar{\theta}^0$ we find that $-\varepsilon\tilde{\theta}^0_{xx} + g(u^0) - g(u^0 + \theta^0) - \lambda\bar{\theta}^0 = 0$. We then similarly obtain (2.8) for this $\tilde{\theta}^0$, and hence we deduce the same conclusion, i.e., $-\theta^0 - \bar{\theta}^0 = \tilde{\theta}^0 \le 0$. This proves the lemma. □

We can also deduce some norm estimates.

Lemma 2.2. *There exists a constant $c > 0$, independent of ε, such that*

$$\|\theta^0(\cdot, y)\|_{H^1_x(0,L_1)} \le c\varepsilon^{-\frac{1}{4}}, \quad \|\theta^0(\cdot, y)\|_{L^2_x(0,L_1)} \le c\varepsilon^{\frac{1}{4}}. \tag{2.9}$$

Proof. The second estimate of (2.9) directly follows from (2.7). To obtain the first estimate, we introduce $\bar{\theta} = -u^0(0,y)e^{-x/\sqrt{\varepsilon}}\delta(x)$, where $\delta(x)$ is a smooth cut-off function with $\delta(x) = 1$ for $x \in [0, L_1/4]$ and $\delta(x) = 0$ for $x \in [3L_1/4, \infty)$. We observe that for a.e. $y \in \mathbb{R}$,

$$\varepsilon \int_0^{L_1} |(\theta^0 - \bar{\theta})_x|^2 \, dx = -\varepsilon \int_0^{L_1} (\theta^0 - \bar{\theta})_{xx}(\theta^0 - \bar{\theta}) \, dx = \int_0^{L_1} ((-g(u^0 + \theta^0) + g(u^0)) + \varepsilon\bar{\theta}_{xx})(\theta^0 - \bar{\theta}) \, dx \le c\varepsilon^{\frac{1}{2}}.$$

where the last inequality follows from the mean value theorem and the L^2-estimate of $\bar{\theta}_{xx}$, $\bar{\theta}$, and θ^0. This implies the lemma. □

Theorem 2.3. *Assume that* (2.4) *holds. Then, there exists a constant* $c > 0$ *such that*

$$\|u^\varepsilon - u^0 - \theta^0\|_\varepsilon \le c\varepsilon. \tag{2.10}$$

Proof. Let $w = u^\varepsilon - u^0 - \theta^0$, then, thanks to (2.5), $w = 0$ on $\partial\Omega$. Subtracting (2.2) and (3.14) from (2.1) we find that

$$\begin{cases} -\varepsilon\Delta w + g(u^\varepsilon) - g(u^0 + \theta^0) = \varepsilon\Delta u^0 + \varepsilon\theta^0_{yy} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Multiplying by w and integrating over Ω we find that

$$\varepsilon \int_\Omega |\nabla w|^2 \, dx\, dy + \int_\Omega (g(u^\varepsilon) - g(u^0 + \theta^0))w \, dx\, dy \le \frac{\lambda}{2} \int_\Omega |w|^2 \, dx\, dy + c\varepsilon^2.$$

Here, the L^2-norm of θ^0_{yy} is derived in Lemma 2.8 below. Thanks to the mean value theorem again and by observing that $(g(u^\varepsilon) - g(u^0 + \theta^0))w \ge \lambda|w|^2$, the theorem is proved. □

We can also obtain the lower bound of $|\theta^0(x,y)|$.

Lemma 2.4. *The corrector* θ^0 *satisfies*

$$|\theta^0(x,y)| \ge |u^0(0,y)| \exp\Big(-\sqrt{\frac{\lambda_0}{\varepsilon}}x\Big) + e.s.t., \tag{2.12}$$

where $\lambda_0 = \max_{|\eta - u^0| \le |\theta^0|} g'(\eta)$.

Proof. From Lemma 2.1 and (2.3), we note that u^0, θ^0 are bounded and hence $\lambda_0 > 0$ is too. We write $\tilde{\theta}^0 = \theta^0 - \bar{\theta}^0$, where $\bar{\theta}^0 = |u^0(0,y)|(\exp(-\sqrt{\lambda_0}x/\sqrt{\varepsilon}) - L_1^{-1}x\exp(-\sqrt{\lambda_0}L_1/\sqrt{\varepsilon})) \ge 0$.

Fixing y, we first prove (2.12) for the case $u^0(0,y) \le 0$. We note that $\tilde{\theta}^0 = 0$ at $x = 0, L_1$. Following the proof of Lemma 2.1, we similarly find that for some η_1 with $|\eta_1 - u^0| < |\theta^0|$,

$$-\varepsilon\tilde{\theta}^0_{xx} + g'(\eta_1)\tilde{\theta}^0 \ge (-g'(\eta_1) + \lambda_0)\bar{\theta}^0 \ge 0.$$

Multiplying by $-\tilde{\theta}^0_- = -\max\{-\tilde{\theta}^0, 0\}$ and integrating over $(0, L_1)$, we obtain

$$\varepsilon \int_0^{L_1} ((\tilde{\theta}^0_-)_x)^2 + \lambda \int_0^{L_1} (\tilde{\theta}^0_-)^2 \le 0.$$

This implies $\tilde{\theta}^0_- = 0$ and hence $|\theta^0| \ge \theta^0 \ge \bar{\theta}^0$, which proves (2.12) for the case $u^0(0,y) \le 0$. For the case $u^0(0,y) > 0$, we write $\tilde{\theta}^0 = -\theta^0 - \bar{\theta}^0$. Then, we similarly deduce that $\tilde{\theta}^0_- = 0$ and hence $|\theta^0| \ge -\theta^0 \ge \bar{\theta}^0$. This proves the lemma. □

Remark 2.5. Thanks to the estimate for θ^0 in L^2, established in Lemma 2.2 with (2.4), Theorem 2.3 implies that

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \le c\varepsilon^{\frac{1}{4}}. \tag{2.13}$$

Furthermore, for $u^0(0,y) \ne 0$ at some $y \in (0, L_2)$ the L^2-norm in (2.13) has a lower bound, i.e., for some $c_0 > 0$,

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \ge c_0\varepsilon^{\frac{1}{4}}.$$

Indeed, from Lemma 2.4 and Theorem 2.3, we find that

$$\|u^\varepsilon - u^0\|_{L^2} \geq \|\theta^0\|_{L^2} - \|u^\varepsilon - u^0 - \theta^0\|_{L^2} \geq c_2\varepsilon^{\frac{1}{4}} - c\varepsilon \geq c_0\varepsilon^{\frac{1}{4}}.$$

2.2 Boundary layer analysis at arbitrary order ε^n, $n \geq 0$

Outer expansion. We now consider the higher order outer expansions $u^\varepsilon \sim \sum_{j=0}^{\infty} \varepsilon^j u^j$. Substituting in (2.1) and using (2.2), we formally write

$$-\varepsilon\Delta\Big(\sum_{j=0}^{\infty} \varepsilon^j u^j\Big) + g\Big(\sum_{j=0}^{\infty} \varepsilon^j u^j\Big) = f. \tag{2.14}$$

Dropping $\mathcal{O}(\varepsilon^{n+1})$ terms, we have

$$-\varepsilon\Delta\Big(\sum_{j=0}^{n-1} \varepsilon^j u^j\Big) + g\Big(\sum_{j=0}^{n} \varepsilon^j u^j\Big) \simeq f.$$

We identify at the order $\mathcal{O}(\varepsilon^j)$, $j = 0, 1, \dots, n$, and find

$$\begin{cases} g(u^0) = f, \\ -\varepsilon^j \Delta u^{j-1} + g\Big(\sum_{k=0}^{j} \varepsilon^k u^k\Big) - g\Big(\sum_{k=0}^{j-1} \varepsilon^k u^k\Big) = 0, \quad j \geq 1. \end{cases} \tag{2.15}$$

We then obtain, e.g.,

$$\begin{aligned} u^0 &= g^{-1}(f), \\ u^1 &= \varepsilon^{-1} g^{-1}(g(u^0) + \varepsilon\Delta u^0) - \varepsilon^{-1}u^0. \end{aligned}$$

More generally, we recursively obtain

$$u^j = \varepsilon^{-j} g^{-1}\Big(g\Big(\sum_{k=0}^{j-1} \varepsilon^k u^k\Big) + \varepsilon^j \Delta u^{j-1}\Big) - \varepsilon^{-j} \sum_{k=0}^{j-1} \varepsilon^k u^k \quad \text{for } j \geq 1. \tag{2.16}$$

To construct the higher order correctors, we assume, for simplicity, that f is infinitely flat at $x = L_1$, i.e.,

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}} = 0 \quad \text{at } x = L_1, \text{ for all } \alpha \geq 0, \tag{2.17}$$

using the multi-index notation

$$\alpha = (\alpha_1, \alpha_2) \quad \text{with } |\alpha| = \alpha_1 + \alpha_2. \tag{2.18}$$

This implies that the u^j, $j \geq 0$, are infinitely flat at $x = L_1$, that is

$$D^\alpha u^j = 0 \quad \text{at } x = L_1, \text{ for all } j, \alpha \geq 0.$$

Thus, we only have boundary layers at $x = 0$ corresponding to u^j.

Correctors. We now proceed with the determination of the correctors. Substituting $u^\varepsilon \sim \sum_{j=0}^{\infty} \varepsilon^j (u^j + \theta^j)$ in (2.1), we have formally

$$-\varepsilon\Delta\Big(\sum_{j=0}^{\infty} \varepsilon^j (u^j + \theta^j)\Big) + g\Big(\sum_{j=0}^{\infty} \varepsilon^j (u^j + \theta^j)\Big) = f. \tag{2.19}$$

We subtract (2.14) from (2.19) to obtain

$$-\varepsilon\Delta\Big(\sum_{j=0}^{\infty} \varepsilon^j \theta^j\Big) + g\Big(\sum_{j=0}^{\infty} \varepsilon^j (u^j + \theta^j)\Big) - g\Big(\sum_{j=0}^{\infty} \varepsilon^j u^j\Big) = 0. \tag{2.20}$$

We first need to handle the nonlinear term to identify the quantities of order ε^j and this is discussed below.

2.3 Treatment of the nonlinear term $g(u)$

In this section, we formally write the nonlinear term $g(\sum_{j=0}^{\infty}\varepsilon^j(u^j+\theta^j)) - g(\sum_{j=0}^{\infty}\varepsilon^j u^j)$ at each order ε^j. Thanks to the Taylor expansion of g about u^0, we have

$$g\Big(\sum_{j=0}^{\infty}\varepsilon^j u^j\Big) = g\Big(u^0 + \sum_{j=1}^{\infty}\varepsilon^j u^j\Big) = \sum_{k=0}^{\infty}\frac{g^{(k)}(u^0)}{k!}\Big(\sum_{j=1}^{\infty}\varepsilon^j u^j\Big)^k.$$

Here, we formally consider $\sum_{j=1}^{\infty}\varepsilon^j u^j = \mathcal{O}(\varepsilon)$. Similarly, expanding at $u^0+\theta^0$, we write

$$g\Big(\sum_{j=0}^{\infty}\varepsilon^j(u^j+\theta^j)\Big) = \sum_{k=0}^{\infty}\frac{g^k(u^0+\theta^0)}{k!}\Big(\sum_{j=1}^{\infty}\varepsilon^j(u^j+\theta^j)\Big)^k.$$

We first observe that

$$\begin{aligned}\Big(\sum_{j=1}^{\infty}\varepsilon^j u^j\Big)^k &= \sum_{|\alpha|=k}\binom{k}{a}((\varepsilon^1u^1)^{\alpha_1}\cdots(\varepsilon^l u^l)^{\alpha_l}\cdots)\\ &= \sum_{|\alpha|=k}\binom{k}{a}((u^1)^{\alpha_1}\cdots(u^l)^{\alpha_l}\cdots)\varepsilon^{(\alpha_1+2\alpha_2+\cdots+l\alpha_l+\cdots)}\\ &= \sum_{|\alpha|=k}\binom{k}{a}u^\alpha\varepsilon^{(\alpha_1+2\alpha_2+\cdots+l\alpha_l+\cdots)},\end{aligned}$$

where

$$u^\alpha = (u^1)^{\alpha_1}\cdots(u^l)^{\alpha_l}\cdots,$$

using the multi-index notation

$$\alpha = (\alpha_1,\dots,\alpha_l,\dots) \quad \text{with } |\alpha| = \alpha_1+\cdots+\alpha_l+\cdots, \tag{2.21}$$

and

$$\binom{k}{\alpha} = \frac{k!}{\alpha_1!\cdots\alpha_l!\cdots}.$$

We similarly find that

$$\Big(\sum_{j=1}^{\infty}\varepsilon^j(u^j+\theta^j)\Big)^k = \sum_{|\alpha|=k}\binom{k}{\alpha}(u+\theta)^\alpha\varepsilon^{(\alpha_1+2\alpha_2+\cdots+l\alpha_l+\cdots)},$$

where

$$(u+\theta)^\alpha = (u^1+\theta^1)^{\alpha_1}\cdots(u^l+\theta^l)^{\alpha_l}\cdots.$$

Hence, we note that

$$\begin{aligned}g\Big(\sum_{j=0}^{\infty}\varepsilon^j(u^j+\theta^j)\Big) - g\Big(\sum_{j=0}^{\infty}\varepsilon^j u^j\Big) &= \sum_{k=0}^{\infty}\frac{1}{k!}\Big[g^{(k)}(u^0+\theta^0)\Big(\sum_{j=1}^{\infty}\varepsilon^j(u^j+\theta^j)\Big)^k - g^{(k)}(u^0)\Big(\sum_{j=1}^{\infty}\varepsilon^j u^j\Big)^k\Big]\\ &= \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\binom{k}{\alpha}\frac{1}{k!}[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha - g^{(k)}(u^0)u^\alpha]\varepsilon^{\alpha_1+2\alpha_2+\cdots+l\alpha_l+\cdots},\end{aligned} \tag{2.22}$$

using the notation (2.21). To arrange the terms at each order ε^j, we set $\alpha_1+2\alpha_2+\cdots+l\alpha_l+\cdots = j$. Since the multi-index α satisfies $|\alpha| = k$, we easily note that $k = |\alpha| \le \alpha_1+2\alpha_2+\cdots+l\alpha_l+\cdots = j$. If one of the α_l with $l \ge j+1$ is greater than or equal to 1, then $\alpha_1+2\alpha_2+\cdots+l\alpha_l+\cdots \ge j+1$, and hence $\alpha_{j+1} = \alpha_{j+2} = \cdots = 0$. Thus, we may write the multi-index notations as

$$\begin{cases}\alpha = (\alpha_1,\dots,\alpha_j), & |\alpha| = \alpha_1+\cdots+\alpha_j,\\ (u+\theta)^\alpha = (u^1+\theta^1)^{\alpha_1}\cdots(u^j+\theta^j)^{\alpha_j}, & u^\alpha = (u^1)^{\alpha_1}\cdots(u^j)^{\alpha_j}.\end{cases} \tag{2.23}$$

Hence, using the multi-index notations in (2.23), we formally write

$$g\Big(\sum_{j=0}^{\infty}\varepsilon^j(u^j+\theta^j)\Big)-g\Big(\sum_{j=0}^{\infty}\varepsilon^j u^j\Big) = \sum_{j=0}^{\infty}\Big\{\sum_{k=0}^{j}\sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+j\alpha_j=j}}\binom{k}{\alpha}\frac{1}{k!}\big[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha-g^{(k)}(u^0)u^\alpha\big]\Big\}\varepsilon^j. \tag{2.24}$$

For the analysis below, we estimate the truncation error corresponding to the expansion (2.24).

Lemma 2.6. *There exists a constant $C>0$, independent of ε, such that*

$$\Big|g\Big(\sum_{j=0}^{n}\varepsilon^j(u^j+\theta^j)\Big)-g\Big(\sum_{j=0}^{n}\varepsilon^j u^j\Big)-G_n\Big|\le C\varepsilon^{n+1},$$

where

$$G_n=\sum_{j=0}^{n}\Big\{\sum_{k=0}^{j}\sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+j\alpha_j=j}}\binom{k}{\alpha}\frac{1}{k!}\big[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha-g^{(k)}(u^0)u^\alpha\big]\Big\}\varepsilon^j,$$

and the multi-index notations are given in (2.23).

Proof. We first note that the G_n given above can be written as

$$G_n=\sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{n}\Big\{\sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+j\alpha_j=j}}\binom{k}{\alpha}\big[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha-g^{(k)}(u^0)u^\alpha\big]\Big\}\varepsilon^j.$$

Thanks to the multinomial theorem, we observe that

$$H_{n,k}:=g^{(k)}(u^0+\theta^0)\Big(\sum_{j=1}^{n}\varepsilon^j(u^j+\theta^j)\Big)^k-g^{(k)}(u^0)\Big(\sum_{j=1}^{n}\varepsilon^j u^j\Big)^k \tag{2.25}$$

$$=\sum_{j=0}^{n}\Big\{\sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+j\alpha_j=j}}\binom{k}{\alpha}\big[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha-g^{(k)}(u^0)u^\alpha\big]\Big\}\varepsilon^j+J_{n,k}, \tag{2.26}$$

where

$$|J_{n,k}|\le C\varepsilon^{n+1}. \tag{2.27}$$

On the other hand, we find from Taylor's theorem that

$$g\Big(\sum_{j=0}^{n}\varepsilon^j(u^j+\theta^j)\Big)-g\Big(\sum_{j=0}^{n}\varepsilon^j u^j\Big)=\sum_{k=0}^{n}\frac{1}{k!}H_{n,k}+R_0=G_n+\sum_{k=0}^{n}\frac{1}{k!}J_{n,k}+R_0,$$

where

$$|R_0|\le\Big|\frac{g^{(n+1)}}{(n+1)!}(\xi_1)\Big(\sum_{j=1}^{n}\varepsilon^j(u^j+\theta^j)\Big)^{n+1}\Big|+\Big|\frac{g^{(n+1)}}{(n+1)!}(\xi_2)\Big(\sum_{j=1}^{n}\varepsilon^j u^j\Big)^{n+1}\Big|.$$

Here, ξ_1 is between $(u^0+\theta^0)$ and $\sum_{j=0}^{n}\varepsilon^j(u^j+\theta^j)$ and ξ_2 is between u^0 and $\sum_{j=0}^{n}\varepsilon^j u^j$. The lemma follows by observing that $|R_0|\le C\varepsilon^{n+1}$. □

We now define the boundary layer correctors θ^j at order $\mathcal{O}(\varepsilon^j)$. From (2.20) and (2.24), using the stretched variable $\bar{x}=x/\sqrt{\varepsilon}$ at each order $\mathcal{O}(\varepsilon^j)$, $j=0,1,\ldots,$ we identify

$$\begin{cases}-\varepsilon\theta^0_{xx}+g(u^0+\theta^0)-g(u^0)=0,\\ -\varepsilon^{j+1}\theta^j_{xx}+\Big\{\sum_{k=1}^{j}\sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+j\alpha_j=j}}\binom{k}{\alpha}\frac{1}{k!}\big[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha-g^{(k)}(u^0)u^\alpha\big]\Big\}\varepsilon^j=\varepsilon^j\theta^{j-1}_{yy}, \quad j\ge 1.\end{cases} \tag{2.28}$$

Dividing by ε^j, rearranging terms in the latter equation at each order $\mathcal{O}(\varepsilon^j)$, and using the fact that

$$|\alpha| = k = 1 \quad \text{and} \quad (\alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j) \Leftrightarrow \alpha_1 = \cdots = \alpha_{j-1} = 0,\ \alpha_j = 1,$$

we rewrite (2.28) as

$$-\varepsilon\theta^0_{xx} + g(u^0 + \theta^0) - g(u^0) = 0,$$
$$-\varepsilon\theta^1_{xx} + g'(u^0 + \theta^0)\theta^1 = -(g'(u^0 + \theta^0) - g'(u^0))u^1 + \theta^0_{yy},$$

and for $j \geq 2$,

$$\begin{aligned} -\varepsilon\theta^j_{xx} + g'(u^0 + \theta^0)\theta^j &= -(g'(u^0 + \theta^0) - g'(u^0))u^j \\ &\quad - \sum_{k=2}^{j} \sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+j\alpha_j=j}} \binom{k}{\alpha} \frac{1}{k!}\big[g^{(k)}(u^0 + \theta^0)(u+\theta)^\alpha - g^{(k)}(u^0)u^\alpha\big] + \theta^{j-1}_{yy}. \end{aligned} \tag{2.29}$$

We supplement the boundary condition on θ^j, for each $j = 0, 1, \ldots$, by

$$\begin{cases} \theta^j = -u^j(0, y) & \text{at } x = 0, \\ \theta^j = 0 & \text{at } x = L_1. \end{cases} \tag{2.30}$$

Remark 2.7. We note that the corrector equations for θ^j, $j \geq 1$, are all linear and this allows us to directly apply the maximum principle. Differentiating the equations in y, the maximum principle also holds for $\frac{\partial^m \theta^j}{\partial y^m}(x, y)$ for $m \geq 1$, $j \geq 0$.

Lemma 2.8. *The correctors θ^j, $j \geq 0$, satisfy*

$$\left|\frac{\partial^m \theta^j}{\partial y^m}(x, y)\right| \leq C \exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big), \quad m \geq 0. \tag{2.31}$$

Proof. We use the maximum principle to prove the lemma. Let $\mathcal{L}$ be the linear operator given by

$$\mathcal{L}u := -\varepsilon u_{xx} + g'(u^0 + \theta^0)u.$$

For $j = 0$, we have

$$-\varepsilon\theta^0_{xx} + g(u^0 + \theta^0) - g(u^0) = 0.$$

We introduce a barrier function $\Psi = C_{01}\exp(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x)$, where C_{01} will be chosen later. We use the mathematical induction on m starting from the case $j = 0$. By (2.7), we already have $|\theta^0(x, y)| \leq c\exp(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x)$. We then see that

$$|\mathcal{L}\theta^0_y| = |-\varepsilon\theta^0_{yxx} + g'(u^0 + \theta^0)\theta^0_y| = |-g'(u^0 + \theta^0)u^0_y + g'(u^0)u^0_y| \leq |g''(\eta)||\theta^0||u^0_y|,$$

by the mean value theorem, for some η between $(u^0 + \theta^0)$ and θ^0. We also find

$$\mathcal{L}\Psi = \Big(g'(u^0 + \theta^0) - \frac{9}{16}\lambda\Big)C_{01}\exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big).$$

Since $g'(u^0 + \theta^0) \geq \lambda$ and $|g''(\eta)||\theta^0||u^0_y|$ is bounded on $\bar{\Omega}$, we can find a positive constant C'_{01} such that

$$|\mathcal{L}\theta^0_y| \leq C'_{01}\exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big) \quad \text{in } \Omega.$$

By the boundary conditions of θ^0, we obtain

$$|\theta^0_y| \leq |u^0(0, y)| \leq C_{01}\exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big) \quad \text{on } \partial\Omega,$$

where $C_{01} = \max(|u^0(0, y)|, C'_{01})$. The maximum principle implies that

$$|\theta^0_y(x, y)| \leq \Psi \quad \text{in } \bar{\Omega}.$$

We suppose by induction that for $k \leq (m-1)$, $m \geq 1$, there exists a positive constant C_{0k} satisfying

$$\left|\frac{\partial^k \theta^0}{\partial y^k}(x,y)\right| \leq C_{0k} \exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big) \quad \text{in } \bar{\Omega}.$$

We then find that

$$\begin{aligned}\mathcal{L}\Big(\frac{\partial^m}{\partial y^m}\theta^0\Big) &= \frac{\partial^{m-1}}{\partial y^{m-1}}(g'(u^0)u^0_y - g'(u^0+\theta^0)u^0_y)\\ &= \sum_{k=0}^{m-1}\Big[h^k_0\Big(g'(u^0),\dots,g^{(m)}(u^0), g'(u^0+\theta^0),\dots,g^{(m)}(u^0+\theta^0), u^0_y,\dots,\frac{\partial^m u^0}{\partial y^m},\\ &\qquad g''(\eta^1),\dots,g^{(m+1)}(\eta^m)\Big)P^k(\theta^0)\Big],\end{aligned} \tag{2.32}$$

where $P^0(\theta^0)=\theta^0$ and $P^k(\theta^0)=\sum_{\alpha_1+\dots+(m-1)\alpha_{m-1}=k}\prod_{i=1}^{m-1}(\partial^i_y\theta^0)^{\alpha_i}$ for $k \geq 1$ with some multivariate polynomials h^k_0 and η^k between $(u^0+\theta^0)$ and u^0. Since g and u^0 are smooth, there exists C'_{0m} such that

$$\mathcal{L}\Big(\frac{\partial^m}{\partial y^m}\theta^0\Big) \leq C'_{0m} \exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big) \quad \text{in } \Omega.$$

We infer from the boundary conditions for θ^0 that

$$\left|\frac{\partial^m}{\partial y^m}\theta^0\right| \leq \left|\frac{\partial^m}{\partial y^m}u^0(0,y)\right| \quad \text{on } \partial\Omega,$$

and using the maximum principle we obtain that

$$\left|\frac{\partial^m}{\partial y^m}\theta^0\right| \leq C_{0m} \exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big) \quad \text{in } \bar{\Omega}, \tag{2.33}$$

where $C_{0m}=\max(|\frac{\partial^m}{\partial y^m}u^0(0,y)|, C'_{0m})$. Similarly, for the case when $j=1$, we see that for $m \geq 0$,

$$\begin{aligned}\mathcal{L}\Big(\frac{\partial^m}{\partial y^m}\theta^1\Big) &= \frac{\partial^m}{\partial y^m}(g'(u^0)u^1 - g'(u^0+\theta^0)u^1 + \theta^0_{yy})\\ &= \sum_{k=0}^{m+2}\Big[h^k_1\Big(g'(u^0),\dots,g^{(m+1)}(u^0), g'(u^0+\theta^0),\dots,g^{(m+1)}(u^0+\theta^0), u^0_y,\dots,\frac{\partial^m u^0}{\partial y^m},\\ &\qquad u^1,\dots,\frac{\partial^m u^1}{\partial y^m}, g''(\eta^1),\dots,g^{m+2}(\eta^{m+1})\Big)P^k(\theta^0)\Big],\end{aligned} \tag{2.34}$$

where $P^0(\theta^0)=\theta^0$ and $P^k(\theta^0)=\sum_{\alpha_1+\dots+(m+2)\alpha_{m+2}=k}\prod_{i=1}^{m+2}(\partial^i_y\theta^0)^{\alpha_i}$ for $k \geq 1$ with for some multivariate polynomial h^k_1 and η^k between $(u^0+\theta^0)$ and u^0. Since g and u^j are smooth and h^k_1 are polynomials, we can find a positive constant C'_{1m}, by the result for θ^0, such that

$$\left|\mathcal{L}\Big(\frac{\partial^m}{\partial y^m}\theta^1\Big)\right| \leq C'_{1m} \exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big) \quad \text{in } \Omega. \tag{2.35}$$

By the boundary condition on θ^1, we also have

$$\left|\frac{\partial^m}{\partial y^m}\theta^1\right| \leq \left|\frac{\partial^m}{\partial y^m}u^1(0,y)\right| \quad \text{on } \partial\Omega. \tag{2.36}$$

We find from (2.35) and (2.36) that for $m \geq 0$,

$$\left|\frac{\partial^m}{\partial y^m}\theta^1\right| \leq C_{1m} \exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big) \quad \text{in } \bar{\Omega},$$

by the maximum principle where $C_{1m}=\max(C'_{1m}, |\frac{\partial^m}{\partial y^m}u^1(0,y)|)$. We now suppose by induction that for $k \leq (j-1)$, $j \geq 1$, there exists a positive constant C_{jm} such that

$$\left|\frac{\partial^m \theta^k}{\partial y^m}(x,y)\right| \leq C_{jm} \exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big), \quad m=0,1,\dots. \tag{2.37}$$

To prove (2.31) at order $k = j$, differentiating (2.29) in y we note that the first and third terms of the right-hand side of (2.29) are similarly estimated as for the case θ^1 by (2.37). We thus estimate the second term there. Observing that for $k \geq 2$,

$$(|\alpha| = k, \quad \alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j) \Leftrightarrow (|\alpha| = k, \quad \alpha_1 + 2\alpha_2 + \cdots + (j-1)\alpha_{(j-1)} = j), \tag{2.38}$$

it suffices to show that for any $m \geq 0$,

$$\left|\frac{\partial^m}{\partial y^m}\Big(\sum_{k=2}^{j} \sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+(j-1)\alpha_{(j-1)}=j}} [g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha - g^{(k)}(u^0)u^\alpha]\Big)\right| \leq C_{jm}\exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}x\Big). \tag{2.39}$$

To prove this, we note that

$$g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha - g^{(k)}(u^0)u^\alpha = (g^{(k)}(u^0+\theta^0) - g^{(k)}(u^0))(u+\theta)^\alpha + g^{(k)}(u^0)((u+\theta)^\alpha - u^\alpha), \tag{2.40}$$

and using the factorization $a^n - b^n = (a-b)\sum_{i=1}^n a^{n-i}b^{i-1}$,

$$(u+\theta)^\alpha - u^\alpha = \sum_{l=1}^{j-1}\Big[\theta^l\Big(\sum_{i=1}^{\alpha_l}(u^l)^{\alpha_l-i}(\theta^l)^{i-1}\Big)\prod_{n=1}^{l-1}(u^n)^{\alpha_n}\prod_{n=l+1}^{j-1}(u^n+\theta^n)^{\alpha_n}\Big],$$

where $\alpha = (\alpha_1, \ldots, \alpha_{j-1})$. Differentiating (2.40) in y, thanks to the mean value theorem, the left-hand side of (2.39) can be written as the sum of the products of θ^k and their derivatives in y for $k \leq (j-1)$. We then conclude, by assumption (2.37), that (2.39) holds true. □

We now estimate, for each $n = 0, 1, \ldots,$ the norm of w_n, where $w_n = u^\varepsilon - \sum_{j=0}^n \varepsilon^j(u^j + \theta^j)$. Summing (2.15) for $j = 0$ to $j = n$, we find

$$-\varepsilon\Delta\Big(\sum_{j=0}^{n-1}\varepsilon^j u^j\Big) + g\Big(\sum_{j=0}^{n}\varepsilon^j u^j\Big) = f. \tag{2.41}$$

Summing (2.28) for $j = 0$ to $j = n$, we find

$$-\varepsilon\sum_{j=0}^n \varepsilon^j\theta^j_{xx} + \sum_{j=0}^n\Big\{\sum_{k=0}^{j}\sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+j\alpha_j=j}}\binom{k}{\alpha}\frac{1}{k!}[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha - g^{(k)}(u^0)u^\alpha]\Big\}\varepsilon^j = \sum_{j=0}^n \varepsilon^j\theta^{j-1}_{yy}.$$

Thanks to Lemma 2.6, this can be written in the form

$$-\varepsilon\Delta\Big(\sum_{j=0}^n\varepsilon^j\theta^j\Big) + g\Big(\sum_{j=0}^n\varepsilon^j(u^j+\theta^j)\Big) - g\Big(\sum_{j=0}^n\varepsilon^j u^j\Big) + R_1 = -\varepsilon^{n+1}\theta^n_{yy} \tag{2.42}$$

with

$$|R_1| \leq c\varepsilon^{n+1}. \tag{2.43}$$

Adding the two above equations (2.41) and (2.42), we find

$$-\varepsilon\Delta\Big(\sum_{j=0}^n\varepsilon^j(u^j+\theta^j)\Big) + g\Big(\sum_{j=0}^n\varepsilon^j(u^j+\theta^j)\Big) = f - \varepsilon^{n+1}\Delta u^n - \varepsilon^{n+1}\theta^n_{yy} - R_1. \tag{2.44}$$

Subtracting (2.44) and from the first equation in (2.1), we find

$$-\varepsilon\Delta w_n + g(u^\varepsilon) - g\Big(\sum_{j=0}^n\varepsilon^j(u^j+\theta^j)\Big) = \varepsilon^{n+1}\Delta u^n + \varepsilon^{n+1}\theta^n_{yy} + R_1. \tag{2.45}$$

We multiply (2.45) by w_n and since $g(u) - g(v) = g'(\xi)(u-v)$ and $g'(\xi) \geq \lambda > 0$, we obtain, by a priori estimate, that

$$\sqrt{\varepsilon}\|w_n\|_{H^1} + \|w_n\|_{L^2} \leq c\varepsilon^{n+1}.$$

We hence proved the following convergence theorem.

Theorem 2.9. *Assume that* (2.17) *holds. Let u^ε be the solution of* (2.1) *and u^j and θ^j be given as in* (2.16) *and* (2.29)–(2.30), *respectively. Then, there exists a positive constant $c > 0$, independent of ε, such that*

$$\Big\|u^\varepsilon - \sum_{j=0}^n\varepsilon^j(u^j+\theta^j)\Big\|_\varepsilon \leq c\,\varepsilon^{n+1}.$$

2.4 Without the assumption (2.17)

If we consider a general smooth function f, i.e., if we remove the assumptions (2.17), we also expect similar boundary layers at $x = L_1$. Let us denote similar boundary layers θ^j at $x = 0$ by θ_l^j, $j \geq 0$, given in (2.29). Similarly, we define the boundary layers at $x = L_1$, denoted by θ_r^j, which satisfy equations (2.29) but with different boundary conditions, i.e.,

$$\begin{cases} \theta_r^j = 0 & \text{at } x = 0, \\ \theta_r^j = -u^j(L_1, y) & \text{at } x = L_1. \end{cases}$$

Then, we define the boundary layers θ^j, $j \geq 0$,

$$\theta^j = \theta_l^j + \theta_r^j. \tag{2.46}$$

Since the corrector equations for θ_l^j, θ_r^j, $j \geq 1$, are linear, we infer from equation (2.42) that

$$-\varepsilon\Delta\Big(\sum_{j=0}^{n} \varepsilon^j \theta^j\Big) + g\Big(\sum_{j=0}^{n} \varepsilon^j (u^j + \theta^j)\Big) - g\Big(\sum_{j=0}^{n} \varepsilon^j u^j\Big) + \widetilde{R}_1 = -\varepsilon^{n+1}\theta_{yy}^n. \tag{2.47}$$

Here,

$$\widetilde{R}_1 = R_1 + R_2, \tag{2.48}$$

where R_1 is given in (2.42)–(2.43) and

$$R_2 = g(u^0 + \theta_l^0) + g(u^0 + \theta_r^0) - g(u^0 + \theta^0) - g(u^0).$$

We now note that R_2 is exponentially small. Indeed, we find that, for all $\alpha \geq 0$,

$$\begin{aligned} \|D^\alpha R_2\|_{L^2(\Omega)} &\leq \|D^\alpha R_2\|_{L^2((0,L_1/2)\times(0,L_2))} + \|D^\alpha R_2\|_{L^2((L_1/2,L_1)\times(0,L_2))} \\ &\leq \big\|D^\alpha\big(g(u^0 + \theta_l^0) + g(u^0 + \text{e.s.t}) - g(u^0 + \theta_l^0 + \text{e.s.t}) - g(u^0)\big)\big\|_{L^2((0,L_1/2)\times(0,L_2))} \\ &\quad + \big\|D^\alpha\big(g(u^0 + \text{e.s.t.}) + g(u^0 + \theta_r^0) - g(u^0 + \text{e.s.t} + \theta_r^0) - g(u^0)\big)\big\|_{L^2((L_1/2,L_1)\times(0,L_2))}, \end{aligned}$$

which is an exponentially small term. Then, the convergence analysis similarly follows as in the above, from which we infer the following theorem.

Theorem 2.10. *Let f be a general smooth function periodic in y with period L_2, let u^ε be the solution of* (2.1), *and let u^j and θ^j be given as in* (2.16) *and* (2.46), *respectively. Then, there exists a positive constant $c > 0$, independent of ε, such that*

$$\Big\|u^\varepsilon - \sum_{j=0}^{n} \varepsilon^j (u^j + \theta^j)\Big\|_\varepsilon \leq c\varepsilon^{n+1}.$$

3 General domains

We now return to the case of a general smooth domain, where equation (1.2) is posed:

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + g(u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{at } \partial\Omega. \end{cases} \tag{3.1}$$

Here, Ω is a general smooth domain, $f = f(x, y)$ and $g(u)$ are given smooth functions with

$$g(0) = 0, \qquad g'(u) \geq \lambda > 0 \quad \text{for all } u \in \mathbb{R}. \tag{3.2}$$

The outer solutions u^j, $j \geq 0$, are the same as in (2.16). That is, in Ω, we have

$$\begin{cases} u^0 = g^{-1}(f), \\ u^1 = \varepsilon^{-1} g^{-1}(g(u^0) + \varepsilon\Delta u^0) - \varepsilon^{-1} u^0, \\ u^j = \varepsilon^{-j} g^{-1}\Big(g\Big(\sum_{k=0}^{j-1} \varepsilon^k u^k\Big) + \varepsilon^j \Delta u^{j-1}\Big) - \varepsilon^{-j} \sum_{k=0}^{j-1} \varepsilon^k u^k \quad \text{for } j \geq 1. \end{cases} \tag{3.3}$$

However, the boundary layers appear in the direction normal to the curved boundary. Thus, the boundary fitted coordinates, i.e., the normal and tangential components along the boundary, are necessary to devise the boundary layer correctors. Here, we consider smooth boundaries $\partial\Omega$ which are parametrised by an arclength η and we assume that $(X(\eta), Y(\eta)) \in \partial\Omega$ is a regular curve, i.e., the tangent vector $T = (X', Y') \neq 0$ for the arclength $0 \leq \eta < L_0$ is measured counterclockwise, where L_0 is the length of the boundary $\partial\Omega$. Hence, we may assume that it has unit speed, i.e., $(X')^2 + (Y')^2 = 1$ (see [34]).

We then define the boundary fitted coordinates:

$$\Omega_{\mathrm{BL}} = \{(x, y) : x = X(\eta) - \xi Y'(\eta),\ y = Y(\eta) + \xi X'(\eta),\ 0 \leq \eta < L_0,\ 0 \leq \xi < \xi_0\}, \tag{3.4}$$

where ξ_0 is the minimum radius of curvature of $\partial\Omega$, i.e., $\xi_0 = 1/\max\kappa(\eta)$. Here we note that $(-Y'(\eta), X'(\eta))$ is the inward unit normal vector to the boundary $\partial\Omega$.

3.1 Boundary fitted coordinates

We introduce the local orthogonal coordinate basis $\mathbf{g}_k$, $i = 1, 2$, on the subdomain Ω_{BL} by setting

$$\mathbf{g}_1 = \frac{\partial x}{\partial \xi}\mathbf{e}_1 + \frac{\partial y}{\partial \xi}\mathbf{e}_2 = -Y'\mathbf{e}_1 + X'\mathbf{e}_2,$$

$$\mathbf{g}_2 = \frac{\partial x}{\partial \eta}\mathbf{e}_1 + \frac{\partial y}{\partial \eta}\mathbf{e}_2 = (X' - \xi Y'')\mathbf{e}_1 + (Y' + \xi X'')\mathbf{e}_2,$$

where $\mathbf{e}_1, \mathbf{e}_2$ is the standard basis in $\mathbb{R}^2$ (see, e.g., [1, 40]). We can easily compute that

$$\mathbf{g}_1 \cdot \mathbf{g}_1 = (X')^2 + (Y')^2 = 1, \tag{3.5}$$

$$\mathbf{g}_1 \cdot \mathbf{g}_2 = \xi X'X'' + \xi Y'Y'' = 0, \tag{3.6}$$

$$\mathbf{g}_2 \cdot \mathbf{g}_2 = 1 + 2\xi(Y'X'' - X'Y'') + \xi^2((X'')^2 + (Y'')^2).$$

Here, we note that differentiating (3.5) implies (3.6). The curvature $\kappa(\eta)$ of $\partial\Omega$ is derived

$$\kappa(\eta) = \frac{X'Y'' - Y'X''}{((X')^2 + (Y')^2)^{\frac{3}{2}}} = X'Y'' - Y'X''$$

and, by the Frenet–Serret relation, $T' = (X'', Y'') = \kappa(\eta)N(\eta)$ where $N(\eta)$ is the unit normal outward vector. Hence, we obtain that

$$\mathbf{g}_2 \cdot \mathbf{g}_2 = (1 - \kappa(\eta)\xi)^2.$$

For the variables (ξ, η) in

$$\Omega_{\xi,\eta} = \{(\xi, \eta) : 0 \leq \eta < L_0,\ 0 \leq \xi < \xi_0\},$$

since $0 \leq \xi < \xi_0 = 1/\max\kappa(\eta)$, we find that $1 - \kappa(\eta)\xi > 0$, and thus we have

$$h_1 := \sqrt{\mathbf{g}_1 \cdot \mathbf{g}_1} = 1, \quad h_2 := \sqrt{\mathbf{g}_2 \cdot \mathbf{g}_2} = 1 - \kappa(\eta)\xi.$$

The gradient and Laplacian operators are then defined as follows:

$$\nabla = \sum_{i=1}^{2} \frac{\mathbf{g}_i}{h_i^2}\frac{\partial}{\partial \zeta^i}, \quad \Delta = \frac{1}{h_1 h_2}\sum_{i=1}^{2} \frac{\partial}{\partial \zeta^i}\Big(\frac{h_1 h_2}{h_i^2}\frac{\partial}{\partial \zeta^i}\Big),$$

where $(\zeta^1, \zeta^2) = (\xi, \eta)$. Hence, we find that

$$\nabla = \Big(-Y'\frac{\partial}{\partial \xi} + (X' - \xi Y'')\sigma^2(\xi, \eta)\frac{\partial}{\partial \eta}, X'\frac{\partial}{\partial \xi} + (Y' + \xi X'')\sigma^2(\xi, \eta)\frac{\partial}{\partial \eta}\Big),$$

and

$$\Delta = \frac{\partial^2}{\partial \xi^2} - \kappa(\eta)\sigma(\xi, \eta)\frac{\partial}{\partial \xi} + \sigma^2(\xi, \eta)\frac{\partial^2}{\partial \eta^2} + \xi\kappa'(\eta)\sigma^3(\xi, \eta)\frac{\partial}{\partial \eta}, \tag{3.7}$$

where

$$\sigma(\xi, \eta) = \frac{1}{1 - \kappa(\eta)\xi}. \tag{3.8}$$

Then, the model equation in (3.1), can be written as

$$-\varepsilon\frac{\partial^2 u^\varepsilon}{\partial \xi^2} + \varepsilon\kappa(\eta)\sigma(\xi,\eta)\frac{\partial u^\varepsilon}{\partial \xi} - \varepsilon\sigma^2(\xi,\eta)\frac{\partial^2 u^\varepsilon}{\partial \eta^2} - \varepsilon\xi\kappa'(\eta)\sigma^3(\xi,\eta)\frac{\partial u^\varepsilon}{\partial \eta} + g(u^\varepsilon) = f. \tag{3.9}$$

Remark 3.1. On the unit circle domain, using polar coordinates, we find $X(\eta) = \cos\eta$, $Y(\eta) = \sin\eta$ and $\sigma(\xi) = (1-\xi)^{-1}$ (note $\kappa(\eta) = 1$). The operators ∇, Δ are simplified to

$$\nabla = \Big(-\cos\eta\frac{\partial}{\partial\xi} - \frac{\sin\eta}{1-\xi}\frac{\partial}{\partial\eta}, -\sin\eta\frac{\partial}{\partial\xi} + \frac{\cos\eta}{1-\xi}\frac{\partial}{\partial\eta}\Big),$$

and

$$\Delta = \frac{1}{(1-\xi)^2}\frac{\partial^2}{\partial\eta^2} - \frac{1}{1-\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}.$$

3.2 Boundary layer analysis at orders ε^0 and $\varepsilon^{\frac{1}{2}}$

Unlike the channel domain (2.1), we will need to introduce a corrector $\theta^{\frac{1}{2}}$ to settle the error from the curvature $\kappa(\eta)$.

In general, to find appropriate asymptotic expansions for the boundary layers, we preform the following expansions near the boundary $\xi = 0$. Subtracting (2.41) from (3.9) and observing that $\sqrt{\varepsilon}$ is the thickness of the boundary layers, as indicated in previous sections, which suggests to use the stretched variable $\bar{\xi} = \xi/\sqrt{\varepsilon}$, we find that

$$\begin{aligned}\Big(-\frac{\partial^2}{\partial\bar{\xi}^2} + \sqrt{\varepsilon}\kappa(\eta)\sigma(\xi,\eta)\frac{\partial}{\partial\bar{\xi}} - \varepsilon\sigma^2(\xi,\eta)\frac{\partial^2}{\partial\eta^2} - \varepsilon\xi\kappa'(\eta)\sigma^3(\xi,\eta)\frac{\partial}{\partial\eta}\Big)\Big(u^\varepsilon - \sum_{j=0}^n \varepsilon^j u^j\Big)\\ + g(u^\varepsilon) - g\Big(\sum_{j=0}^n \varepsilon^j u^j\Big) = \varepsilon^{n+1}\Delta u^n.\end{aligned} \tag{3.10}$$

To address the terms at all the orders of ε in the boundary layers, we have to resolve the effect of curvature $\kappa(\eta)$. Here, $\kappa(\eta) = 0$ for the channel domain. We first observe that $\sigma(\xi,\eta) = (1-\kappa(\eta))^{-1} = \sum_{l=0}^{\infty}(\kappa(\eta)\bar{\xi})^l\varepsilon^{\frac{l}{2}}$ and the powers of $\sigma(\xi,\eta)$ can be similarly expressed (see also (3.38) below). We then take into account the mean value theorem for the first two terms in the second line of (3.10), i.e., $g(u^\varepsilon) - g(\sum_{j=0}^n \varepsilon^j u^j) = g'(u)(u^\varepsilon - \sum_{j=0}^n \varepsilon^j u^j)$ for some u. In this way, we can balance the difference between u^ε and the outer expansion at order n by using the inner expansion near $\partial\Omega$ in the form

$$u^\varepsilon - \sum_{j=0}^n \varepsilon^j u^j \simeq \sum_{j=0}^n (\varepsilon\theta^j + \varepsilon^{j+\frac{1}{2}}\theta^{j+\frac{1}{2}}) \quad \text{near } \partial\Omega. \tag{3.11}$$

By comparing the terms of the same order ε^j on $\partial\Omega$, we deduce from (3.11) the following boundary conditions for $j \geq 0$:

$$\begin{cases}\theta^j = -u^j & \text{on } \partial\Omega,\\ \theta^{j+\frac{1}{2}} = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.12}$$

We now find two leading order correctors θ^0 and $\theta^{\frac{1}{2}}$ satisfying

$$-\varepsilon(\theta^0_{\xi\xi} + \sqrt{\varepsilon}\theta^{\frac{1}{2}}_{\xi\xi}) + \varepsilon\kappa(\eta)(\theta^0_\xi + \sqrt{\varepsilon}\theta^{\frac{1}{2}}_\xi) + g(u^0 + \theta^0 + \sqrt{\varepsilon}\theta^{\frac{1}{2}}) - g(u^0) \simeq 0. \tag{3.13}$$

By the Taylor expansion, dropping smaller terms and using the stretched variable $\bar{\xi} = \xi/\sqrt{\varepsilon}$, we obtain $\bar{\theta}^0$ at order $\mathcal{O}(\varepsilon^0)$, which is a solution of

$$-\bar{\theta}^0_{\bar{\xi}\bar{\xi}} + g(u^0 + \bar{\theta}^0) - g(u^0) = 0, \tag{3.14}$$

and from (3.13) at order $\mathcal{O}(\varepsilon^{\frac{1}{2}})$, we again find $\bar{\theta}^{\frac{1}{2}}$ such that

$$-\bar{\theta}^{\frac{1}{2}}_{\bar{\xi}\bar{\xi}} + g'(u^0 + \bar{\theta}^0)\bar{\theta}^{\frac{1}{2}} = -\kappa(\eta)\bar{\theta}^0_{\bar{\xi}}. \tag{3.15}$$

These equations are supplemented with the respective boundary conditions, i.e.,

$$\begin{cases} \bar{\theta}^0 = -u^0|_{\xi=0} = -u^0(X(\eta), Y(\eta)) & \text{at } \xi = 0, \\ \bar{\theta}^0 = 0 & \text{at } \xi = \xi_0, \end{cases} \qquad \begin{cases} \bar{\theta}^{\frac{1}{2}} = 0 & \text{at } \xi = 0, \\ \bar{\theta}^{\frac{1}{2}} = 0 & \text{at } \xi = \xi_0. \end{cases} \tag{3.16}$$

Then, we extend $\bar{\theta}^0$ and $\bar{\theta}^{\frac{1}{2}}$ by zero for $\xi > \xi_0$ and these extensions are still denoted by the same notations. However, these extensions are not smooth. Thus, for the analysis below, let us define

$$\theta^0 = \bar{\theta}^0 \delta(\xi), \quad \theta^{\frac{1}{2}} = \bar{\theta}^{\frac{1}{2}} \delta(\xi),$$

where $\delta(\xi)$ is a smooth cut-off function given by

$$\delta(\xi) = \begin{cases} 1, & \xi \in [0, \xi_0/4], \\ 0, & \xi \in [3\xi_0/4, \infty). \end{cases} \tag{3.17}$$

Remark 3.2. We obtain θ^0 by multiplying $\bar{\theta}^0$ by the cut-off function. This allows us to use the same estimates for θ^0 as $\bar{\theta}^0$. We proceed similarly for $\theta^{\frac{1}{2}}$.

Lemma 3.3. *The following pointwise estimate holds for $\bar{\theta}^0$:*

$$|\bar{\theta}^0| \le |u^0|_{\xi=0} \exp\Big(-\sqrt{\frac{\lambda}{\varepsilon}}\xi\Big). \tag{3.18}$$

Furthermore, the derivatives of $\bar{\theta}^0$ satisfy pointwise, for $l, m, n \ge 0$, the following:

$$\Big|\xi^n \frac{\partial^{l+m}\bar{\theta}^0}{\partial \xi^l \partial \eta^m}\Big| \le c\varepsilon^{\frac{n-l}{2}} \exp\Big(-\frac{1}{2}\sqrt{\frac{\lambda}{\varepsilon}}\xi\Big). \tag{3.19}$$

Proof. We first set $\psi = |u^0|_{\xi=0} \exp(-\sqrt{\lambda}\bar{\xi})$. Writing $\tilde{\theta} = \bar{\theta}^0 - \psi$, we deduce from (3.14) that

$$-\tilde{\theta}_{\bar{\xi}\bar{\xi}} + g(u^0 + \bar{\theta}^0) - g(u^0) - \lambda\psi = 0,$$

which implies

$$-\tilde{\theta}_{\bar{\xi}\bar{\xi}} + g'(\eta)\tilde{\theta} \le 0. \tag{3.20}$$

Thanks to (3.16), multiplying (3.20) by $\tilde{\theta}_+(\bar{\xi}, \cdot) \in H_0^1(0, \infty)$ and integrating over $(0, \infty)$, we find that

$$\int_0^\infty |(\tilde{\theta}_+)_{\bar{\xi}}|^2 \, d\bar{\xi} + \lambda \int_0^\infty |\tilde{\theta}_+|^2 \, d\bar{\xi} \le 0.$$

This implies $\tilde{\theta}_+ = 0$ for all $\bar{\xi} \ge 0$ and thus $\bar{\theta}^0 \le \psi$. The other inequality $-\bar{\theta}^0 \le \psi$ similarly follows.

We now find the estimates for the derivatives as in (3.19). For $l = 0$ and $n = 0$, it immediately follows, from Lemma 2.8, that

$$\Big|\frac{\partial^m \bar{\theta}^0}{\partial \eta^m}\Big| \le c \exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}\xi\Big).$$

Multiplying by ξ^n, the estimate (3.19) follows for $l = 0$ and $n \ge 0$.

Then, at higher orders l, we use the multi-index $\alpha = (l-2, m)$ with $l \ge 2$ and $D^\alpha = \partial_{\bar{\xi}}^{l-2}\partial_\eta^m$. Applying the operator D^α to (3.14), we have

$$\partial_{\bar{\xi}}^l \partial_\eta^m \bar{\theta}^0 = (D^\alpha \bar{\theta}^0)_{\bar{\xi}\bar{\xi}} = D^\alpha(g(u^0 + \bar{\theta}^0) - g(u^0)).$$

Thanks to the mean value theorem, we observe that

$$D^\alpha(g(u^0 + \bar{\theta}^0) - g(u^0)) = \text{finite sum of products of } D^\beta \bar{\theta}^0 \text{ with } \beta \le (l-2, m), \tag{3.21}$$

and we can thus inductively prove (3.19) for any $l \ge 0$ as long as the case for $l = 1$ is proved.

For $l = 1$, we let $\mu > 0$, which will be determined. We infer from (3.14) that

$$(\bar{\theta}^0_{\bar{\xi}} e^{\mu\bar{\xi}})_{\bar{\xi}} = (g(u^0 + \bar{\theta}^0) - g(u^0))e^{\mu\bar{\xi}} + \mu\bar{\theta}^0_{\bar{\xi}} e^{\mu\bar{\xi}} = (g(u^0 + \bar{\theta}^0) - g(u^0))e^{\mu\bar{\xi}} + (\mu\bar{\theta}^0 e^{\mu\bar{\xi}})_{\bar{\xi}} - \mu^2\bar{\theta}^0 e^{\mu\bar{\xi}}.$$

Integrating over $(\bar{\xi}, \infty)$, we find

$$C - \bar{\theta}^0_{\bar{\xi}} e^{\mu\bar{\xi}} = I_1(\bar{\xi}) - \mu\bar{\theta}^0 e^{\mu\bar{\xi}},$$

where

$$I_1(\bar{\xi}) = \int_{\bar{\xi}}^{\infty} (g(u^0 + \bar{\theta}^0) - g(u^0) - \mu^2\bar{\theta}^0)(t, \eta)e^{\mu t}\, \mathrm{d}t.$$

Then,

$$\bar{\theta}^0_{\bar{\xi}} = Ce^{-\mu\bar{\xi}} - I_1(\bar{\xi})e^{-\mu\bar{\xi}} + \mu\bar{\theta}^0, \tag{3.22}$$

and integrating over $(\bar{\xi}, \infty)$, we find that

$$-\bar{\theta}^0 = D + \mu^{-1}Ce^{-\mu\bar{\xi}} - \int_{\bar{\xi}}^{\infty} I_1(s)e^{-\mu s}\, \mathrm{d}s + \mu \int_{\bar{\xi}}^{\infty} \bar{\theta}^0(s, \eta)\, \mathrm{d}s. \tag{3.23}$$

Let $\mu = p\sqrt{\lambda}$, where p, a constant independent of ε, is to be determined. For $0 < p < 1$, since (3.18) yields $|\bar{\theta}^0(t, \eta)| \le c\exp(-\sqrt{\lambda}t)$, we find that

$$|I_1(\bar{\xi})| \le c \int_{\bar{\xi}}^{\infty} e^{-\sqrt{\lambda}(1-p)t}\, \mathrm{d}t \le ce^{-\sqrt{\lambda}(1-p)\bar{\xi}},$$

and

$$\Big| \int_{\bar{\xi}}^{\infty} I_1(s)e^{-\mu s}\, \mathrm{d}s \Big| \le c \int_{\bar{\xi}}^{\infty} e^{-\sqrt{\lambda}s}\, \mathrm{d}s \le ce^{-\sqrt{\lambda}\bar{\xi}}.$$

Let us choose $p = \frac{3}{4}$ and thus $\mu = \frac{3}{4}\sqrt{\lambda}$. Applying the boundary conditions to (3.23), i.e., $\bar{\theta}^0 \to 0$ as $\bar{\xi} \to \infty$ and $\bar{\theta}^0 = -u^0|_{\xi=0}$ at $\bar{\xi} = 0$, we find that $D = 0$ and that

$$-u^0(0, \eta) = \bar{\theta}^0(0, \eta) = -\mu^{-1}C + \int_0^{\infty} I_1(s)e^{-\mu s}\, \mathrm{d}s - \mu \int_0^{\infty} \bar{\theta}^0(s, \eta)\, \mathrm{d}s.$$

Thus,

$$C = \mu\Big[u^0(0, \eta) + \int_0^{\infty} I_1(s)e^{-\mu s}\, \mathrm{d}s - \mu \int_0^{\infty} \bar{\theta}^0(s, \eta)\, \mathrm{d}s\Big].$$

To estimate $\partial_\eta^m \bar{\theta}^0_{\bar{\xi}}$, we apply ∂_η^m to (3.22) and find that

$$\begin{aligned} |\partial_\eta^m \bar{\theta}^0_{\bar{\xi}}| &\le (|\partial_\eta^m C| + |\partial_\eta^m I_1(\bar{\xi})|)\exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}\xi\Big) + c|\partial_\eta^m \bar{\theta}^0| \\ &\le c\exp\Big(-\frac{3}{4}\sqrt{\frac{\lambda}{\varepsilon}}\xi\Big). \end{aligned} \tag{3.24}$$

Here, from (3.21) with $\alpha = (0, m)$, i.e., $l = 2$, and (3.19) with $l = 0$, we used the fact that

$$|\partial_\eta^m C| + |\partial_\eta^m I_1(\bar{\xi})| \le c. \qquad \square$$

We similarly derive the pointwise estimate for $\bar{\theta}^{\frac{1}{2}}$ and the result appears in the following lemma.

Lemma 3.4. *The corrector $\bar{\theta}^{\frac{1}{2}}$ satisfies pointwise, for $l, m, n \geq 0$, the following:*

$$\left|\xi^n \frac{\partial^{l+m}\bar{\theta}^{\frac{1}{2}}}{\partial\xi^l\partial\eta^m}\right| \leq c\varepsilon^{\frac{n-l}{2}} \exp\left(-\frac{1}{2}\sqrt{\frac{\lambda}{\varepsilon}}\xi\right). \tag{3.25}$$

Proof. Let $\mathcal{L}$ be the linear operator given by

$$\mathcal{L}u = -u_{\bar{\xi}\bar{\xi}} + g'(u^0 + \bar{\theta}^0)u.$$

We repeat the same argument as in the proof of Lemma 3.3. For $l = 0$, the estimates (3.25) hold true by the maximum principle applied to $\mathcal{L}(\partial_\eta^m \bar{\theta}^{\frac{1}{2}})$ with (3.15). For $l = 1$, we infer from (3.15) that

$$(\bar{\theta}^{\frac{1}{2}}_{\bar{\xi}} e^{\mu\bar{\xi}})_{\bar{\xi}} = g'(u^0 + \bar{\theta}^0)\bar{\theta}^{\frac{1}{2}} e^{\mu\bar{\xi}} + \kappa(\eta)\bar{\theta}^0_{\bar{\xi}} e^{\mu\bar{\xi}} + (\mu\bar{\theta}^{\frac{1}{2}} e^{\mu\bar{\xi}})_{\bar{\xi}} - \mu^2\bar{\theta}^{\frac{1}{2}} e^{\mu\bar{\xi}}.$$

We integrate over $(\bar{\xi}, \infty)$, to find that

$$C - \bar{\theta}^{\frac{1}{2}}_{\bar{\xi}} e^{\mu\bar{\xi}} = I_2(\bar{\xi}) - \mu\bar{\theta}^{\frac{1}{2}} e^{\mu\bar{\xi}},$$

where

$$I_2(\bar{\xi}) = \int_{\bar{\xi}}^{\infty} (g'(u^0 + \bar{\theta}^0)\bar{\theta}^{\frac{1}{2}} + \kappa(\eta)\bar{\theta}^0_{\bar{\xi}} - \mu^2\bar{\theta}^{\frac{1}{2}})(t, \eta)e^{\mu t}\,\mathrm{d}t.$$

Let $\mu = p\sqrt{\lambda}$, where p is a constant independent of ε to be chosen later. From estimates (3.19), we note that $|\bar{\theta}^0_{\bar{\xi}}(t, \eta)| \leq c\exp(-\sqrt{\lambda}t/2)$. Using the same argument as in the proof of Lemma 3.3, we only need to show that for $0 < p < 1$,

$$|I_2(\bar{\xi})| \leq c\int_{\bar{\xi}}^{\infty} e^{-\sqrt{\lambda}(1-p)t}\,\mathrm{d}t \leq ce^{-\sqrt{\lambda}(1-p)\bar{\xi}}. \tag{3.26}$$

Following then the same procedure as in the proof of Lemma 3.3 with the boundary conditions (3.16), we can obtain (3.24) for $\bar{\theta}^{\frac{1}{2}}$.

For $l \geq 2$, differentiating (3.15) in $\bar{\xi}$ and using estimates (3.19), the lemma is inductively proved. □

We now find the norm estimate in L^2 and H^1 for $\bar{\theta}^0$ and $\bar{\theta}^{\frac{1}{2}}$ in the next lemma.

Lemma 3.5. *Let $l, m, n \geq 0$. Then, there exists $c > 0$ such that*

$$\left\|\xi^n \frac{\partial^{l+m}\bar{\theta}^0}{\partial\xi^l\partial\eta^m}\right\|_{L^2_\xi(\Omega)} + \left\|\xi^n \frac{\partial^{l+m}\bar{\theta}^{\frac{1}{2}}}{\partial\xi^l\partial\eta^m}\right\|_{L^2_\xi(\Omega)} \leq c\varepsilon^{\frac{n-l}{2}+\frac{1}{4}}. \tag{3.27}$$

Proof. We infer from Lemma 3.3 and Lemma 3.4 that (3.27) holds. □

We now introduce the analogue of Theorem 2.3.

Theorem 3.6. *Assume f is a general smooth function and Ω is a general smooth domain. Then, there exists a positive constant $c > 0$ such that*

$$\|u^\varepsilon - u^0 - \theta^0\|_\varepsilon \leq c\varepsilon^{\frac{3}{4}}.$$

Proof. We set $w = u^\varepsilon - u^0 - \theta^0$. To avoid the singularity of $\sigma(\xi, \eta) = (1 - \kappa(\eta)\xi)^{-1}$, we use the smooth cut-off function $\tilde{\delta}(\xi)$ such that

$$\tilde{\delta}(\xi) = \begin{cases} 1, & \xi \in [0, \xi_0/2], \\ 0, & \xi \in [3\xi_0/4, \infty). \end{cases} \tag{3.28}$$

Here, we recall that $\xi_0 = 1/\max\kappa(\eta)$. Noting that $\theta^0 = 0$ for $\xi \geq \xi_0/2$ and denoting by $(\cdot\,,\cdot)$ the scalar product in the space $L^2(\Omega)$, we can write

$$\begin{aligned}(\varepsilon\Delta\theta^0 - g(u^0 + \theta^0) + g(u^0), w) &= (\varepsilon\Delta\theta^0 - g(u^0 + \theta^0) + g(u^0), \tilde{\delta}(\xi)w) \\ &= (G_0 + E_0, \tilde{\delta}(\xi)w) + (\varepsilon\Delta\bar{\theta}^0 - g(u^0 + \bar{\theta}^0) + g(u^0), \tilde{\delta}(\xi)w) \\ &= ((G_0 + E_0)\tilde{\delta}(\xi), w) + (R_0\tilde{\delta}(\xi), w),\end{aligned} \tag{3.29}$$

thanks to (3.14). Here,

$$E_0 = \varepsilon\Delta(\theta^0 - \bar{\theta}^0),$$
$$G_0 = g(u^0 + \bar{\theta}^0) - g(u^0 + \theta^0),$$
$$R_0 = -\varepsilon\kappa(\eta)\sigma(\xi,\eta)\frac{\partial\bar{\theta}^0}{\partial\xi} + \varepsilon\sigma^2(\xi,\eta)\frac{\partial^2\bar{\theta}^0}{\partial\eta^2} + \varepsilon\xi\kappa'(\eta)\sigma^3(\xi,\eta)\frac{\partial\bar{\theta}^0}{\partial\eta}.$$

Since $\theta^0 - \bar{\theta}^0 = \bar{\theta}^0(1-\delta(\xi))$, it is very easy to prove that G_0 and E_0 are e.s.t. We also find that

$$\|R_0\tilde{\delta}(\xi)\|_{L^2(\Omega)} \le c\varepsilon^{\frac{3}{4}}.$$

Taking the inner product of (3.3) and (3.1), respectively, with w, we write

$$(-\varepsilon\Delta u^\varepsilon + g(u^\varepsilon), w) = (f, w), \tag{3.30}$$
$$(g(u^0), w) = (f, w). \tag{3.31}$$

From (3.29), (3.30) and (3.31), we find that

$$\begin{aligned}
&(-\varepsilon\Delta w + g(u^\varepsilon) - g(u^0 + \theta^0), w)\\
&\quad = (-\varepsilon\Delta u^\varepsilon + g(u^\varepsilon), w) - (g(u^0), w) + (\varepsilon\Delta u^0, w) + (\varepsilon\Delta\theta^0 - g(u^0+\theta^0) + g(u^0), w)\\
&\quad = (\varepsilon\Delta u^0, w) + (G_0\tilde{\delta}(\xi), w) + (R_0\tilde{\delta}(\xi), w)\\
&\quad \le c\varepsilon^{\frac{3}{4}}\|w\|_{L^2(\Omega)}.
\end{aligned}$$

Thanks to (3.2), this completes the proof of Theorem 3.6. □

Theorem 3.7. *Assume that f is a general smooth function and Ω is a general smooth domain. Then, there exists a positive constant $c > 0$ such that*

$$\|u^\varepsilon - u^0 - \theta^0 - \varepsilon^{\frac{1}{2}}\theta^{\frac{1}{2}}\|_\varepsilon \le c\varepsilon.$$

Proof. We define $w^{\frac{1}{2}} = u^\varepsilon - u^0 - \theta^0 - \sqrt{\varepsilon}\theta^{\frac{1}{2}}$, then we find

$$\begin{aligned}
&(\varepsilon\Delta(\theta^0 + \sqrt{\varepsilon}\theta^{\frac{1}{2}}) - g(u^0 + \theta^0 + \sqrt{\varepsilon}\theta^{\frac{1}{2}}) + g(u^0), w^{\frac{1}{2}})\\
&\quad = (\varepsilon\Delta(\theta^0 + \sqrt{\varepsilon}\theta^{\frac{1}{2}}) - g(u^0 + \theta^0 + \sqrt{\varepsilon}\theta^{\frac{1}{2}}) + g(u^0), \tilde{\delta}(\xi)w^{\frac{1}{2}})\\
&\quad = (G_{\frac{1}{2}} + E_{\frac{1}{2}}, \tilde{\delta}(\xi)w^{\frac{1}{2}}) + (\varepsilon\Delta(\bar{\theta}^0 + \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}}) - g(u^0 + \bar{\theta}^0 + \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}}) + g(u^0), \tilde{\delta}(\xi)w^{\frac{1}{2}})\\
&\quad = ((G_{\frac{1}{2}} + E_{\frac{1}{2}})\tilde{\delta}(\xi), w^{\frac{1}{2}}) + (R_{\frac{1}{2}}\tilde{\delta}(\xi), w^{\frac{1}{2}}),
\end{aligned} \tag{3.32}$$

by (3.14) and by (3.15), where

$$\begin{cases}
E_{\frac{1}{2}} = \varepsilon\Delta(\theta^0 + \sqrt{\varepsilon}\theta^{\frac{1}{2}} - \bar{\theta}^0 - \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}}),\\
G_{\frac{1}{2}} = g(u^0 + \bar{\theta}^0 + \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}}) - g(u^0 + \theta^0 + \sqrt{\varepsilon}\theta^{\frac{1}{2}}),\\
R_{\frac{1}{2}} = -\varepsilon\kappa(\eta)\sigma(\xi,\eta)\dfrac{\partial}{\partial\xi}(\bar{\theta}^0 + \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}}) + \varepsilon\kappa(\eta)\dfrac{\partial}{\partial\xi}\bar{\theta}^0\\
\qquad + \varepsilon\sigma^2(\xi,\eta)\dfrac{\partial^2}{\partial\eta^2}(\bar{\theta}^0 + \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}}) + \varepsilon\xi\kappa'(\eta)\sigma^3(\xi,\eta)\dfrac{\partial}{\partial\eta}(\bar{\theta}^0 + \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}})\\
\qquad + g(u^0 + \bar{\theta}^0) + g'(u^0 + \bar{\theta}^0)\sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}} - g(u^0 + \bar{\theta}^0 + \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}}).
\end{cases} \tag{3.33}$$

We note that thanks to the Taylor expansion, we have

$$g(u^0 + \bar{\theta}^0 + \sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}}) - g(u^0 + \bar{\theta}^0) = g'(u^0 + \bar{\theta}^0)\sqrt{\varepsilon}\bar{\theta}^{\frac{1}{2}} + T_{\frac{1}{2}},$$

where

$$\|T_{\frac{1}{2}}\|_{L^2(\Omega)} \le c\|\varepsilon(\bar{\theta}^{\frac{1}{2}})^2\|_{L^2(\Omega)} \le c\varepsilon^{\frac{5}{4}}. \tag{3.34}$$

On the other hand,

$$\begin{aligned}&\Big\|-\varepsilon\kappa(\eta)\sigma(\xi,\eta)\tilde\delta(\xi)\frac{\partial}{\partial\xi}(\bar\theta^0+\sqrt{\varepsilon}\bar\theta^{\frac12})+\varepsilon\kappa(\eta)\tilde\delta(\xi)\frac{\partial}{\partial\xi}\bar\theta^0\Big\|_{L^2(\Omega)}\\&=\Big\|-\varepsilon\sqrt{\varepsilon}\kappa(\eta)\frac{\partial}{\partial\xi}\bar\theta^{\frac12}\tilde\delta(\xi)-\varepsilon\kappa(\eta)\sum_{j=1}^{\infty}(\kappa(\eta)\sqrt{\varepsilon}\bar\xi)^j\tilde\delta(\xi)\frac{\partial}{\partial\xi}(\bar\theta^0+\sqrt{\varepsilon}\bar\theta^{\frac12})\Big\|_{L^2(\Omega)}\\&\le c\varepsilon^{\frac54},\end{aligned}\tag{3.35}$$

by Remark 3.2 and (3.27). We infer from Lemma 3.4, (3.34) and (3.35) that

$$\|R_{\frac12}\tilde\delta(\xi)\|_{L^2(\Omega)}\le c\varepsilon^{\frac54},$$

while $E_{\frac12}$ and $G_{\frac12}$ are e.s.t. We now find from (3.32) that

$$\begin{aligned}&(-\varepsilon\Delta w^{\frac12}+g(u^\varepsilon)-g(u^0+\theta^0+\sqrt{\varepsilon}\theta^{\frac12}),w^{\frac12})\\&\quad=(-\varepsilon\Delta u^\varepsilon+g(u^\varepsilon)-g(u^0),w^{\frac12})+(\varepsilon\Delta u^0,w^{\frac12})+(\varepsilon\Delta(\theta^0+\sqrt{\varepsilon}\theta^{\frac12})-g(u^0+\theta^0+\sqrt{\varepsilon}\theta^{\frac12})+g(u^0),\tilde\delta(\xi)w^{\frac12})\\&\quad=(\varepsilon\Delta u^0,w^{\frac12})+((G_{\frac12}+E_{\frac12})\tilde\delta(\xi),w^{\frac12})+(R_{\frac12}\tilde\delta(\xi),w^{\frac12}).\end{aligned}\tag{3.36}$$

We then find from (3.36) and the mean value theorem that

$$(-\varepsilon\Delta w^{\frac12}+g'(\zeta)w^{\frac12},w^{\frac12})=(\varepsilon\Delta u^0,w^{\frac12})+((G_{\frac12}+E_{\frac12})\tilde\delta(\xi),w^{\frac12})+(R_{\frac12}\tilde\delta(\xi),w^{\frac12})$$

for some ζ between u^ε and $(u^0+\theta^0+\sqrt{\varepsilon}\theta^{\frac12})$. We then conclude

$$(-\varepsilon\Delta w^{\frac12}+g'(\zeta)w^{\frac12},w^{\frac12})\le c\varepsilon\|w^{\frac12}\|_{L^2(\Omega)},$$

which implies

$$\varepsilon\|w^{\frac12}\|^2_{H^1(\Omega)}+\lambda\|w^{\frac12}\|^2_{L^2(\Omega)}\le c\,\varepsilon^2+\frac{\lambda}{2}\|w^{\frac12}\|^2_{L^2(\Omega)}.$$ □

3.3 Boundary layer analysis at arbitrary orders ε^n and $\varepsilon^{n+\frac12}$, $n\ge 0$

Similarly as in (2.20), we formally write

$$-\varepsilon\Delta\Big(\sum_{j=0}^{\infty}\varepsilon^j(\theta^j+\sqrt{\varepsilon}\theta^{j+\frac12})\Big)+g\Big(\sum_{j=0}^{\infty}\varepsilon^j(u^j+\theta^j+\sqrt{\varepsilon}\theta^{j+\frac12})\Big)-g\Big(\sum_{j=0}^{\infty}\varepsilon^ju^j\Big)=0.\tag{3.37}$$

Because of the one-dimensional nature of these boundary layers near the boundary in the direction normal to the boundary, we now introduce the boundary fitted coordinates. We transform the Laplacian Δ as in (3.7) and (3.8).

Using the geometric series expansions

$$(1-r)^{-1}=\sum_{l=0}^{\infty}r^l,\quad(1-r)^{-2}=\sum_{l=0}^{\infty}(l+1)r^l\quad\text{and}\quad(1-r)^{-3}=\frac12\sum_{l=0}^{\infty}(l+1)(l+2)r^l,$$

we formally write, e.g.,

$$\sigma^3(\xi,\eta)=(1-\kappa(\eta)\xi)^{-3}=\frac12\sum_{l=0}^{\infty}(l+1)(l+2)(\kappa(\eta)\bar\xi)^l\varepsilon^{\frac l2}.\tag{3.38}$$

Then,

$$\begin{aligned}\varepsilon\Delta\Big(\sum_{j=0}^{\infty}(\varepsilon^j\bar\theta^j+\varepsilon^{j+\frac12}\bar\theta^{j+\frac12})\Big)&=\frac{\partial^2}{\partial\bar\xi^2}\Big(\sum_{j=0}^{\infty}(\varepsilon^j\bar\theta^j+\varepsilon^{j+\frac12}\bar\theta^{j+\frac12})\Big)-\kappa(\eta)\Big(\sum_{l=0}^{\infty}(\kappa(\eta)\bar\xi)^l\varepsilon^{\frac l2}\Big)\frac{\partial}{\partial\bar\xi}\Big(\sum_{j=0}^{\infty}(\varepsilon^{j+\frac12}\bar\theta^j+\varepsilon^{j+1}\bar\theta^{j+\frac12})\Big)\\&\quad+\Big(\sum_{l=0}^{\infty}(l+1)(\kappa(\eta)\bar\xi)^l\varepsilon^{\frac l2}\Big)\frac{\partial^2}{\partial\eta^2}\Big(\sum_{j=0}^{\infty}(\varepsilon^{j+1}\bar\theta^j+\varepsilon^{j+\frac32}\bar\theta^{j+\frac12})\Big)\\&\quad+\frac{\bar\xi}{2}\kappa'(\eta)\Big(\sum_{l=0}^{\infty}(l+1)(l+2)(\kappa(\eta)\bar\xi)^l\varepsilon^{\frac l2}\Big)\frac{\partial}{\partial\eta}\Big(\sum_{j=0}^{\infty}(\varepsilon^{j+\frac32}\bar\theta^j+\varepsilon^{j+2}\bar\theta^{j+\frac12})\Big)\\&=\frac{\partial^2}{\partial\bar\xi^2}\Big(\sum_{j=0}^{\infty}(\varepsilon^j\bar\theta^j+\varepsilon^{j+\frac12}\bar\theta^{j+\frac12})\Big)+I_1^\infty+I_2^\infty+I_3^\infty.\end{aligned}\tag{3.39}$$

Using

$$\sum_{l=0}^{\infty}(\kappa(\eta)\bar{\xi})^l\varepsilon^{\frac{l}{2}} = \sum_{m=0}^{\infty}(\kappa(\eta)\bar{\xi})^{2m}\varepsilon^m + \sum_{m=0}^{\infty}(\kappa(\eta)\bar{\xi})^{2m+1}\varepsilon^{m+\frac{1}{2}},$$

we rearrange the terms according to the order of ε and then we find the last expression in (3.39). The I_k are defined and expanded as follows

$$\begin{cases} I_1^n = -\kappa(\eta)\sum_{j=0}^{n}(\varepsilon^j J_1^j(\bar{\theta}) + \varepsilon^{j+\frac{1}{2}}J_1^{j+\frac{1}{2}}(\bar{\theta})), \\ I_2^n = \sum_{j=0}^{n}(\varepsilon^j J_2^j(\bar{\theta}) + \varepsilon^{j+\frac{1}{2}}J_2^{j+\frac{1}{2}}(\bar{\theta})), \\ I_3^n = \frac{\bar{\xi}}{2}\kappa'(\eta)\sum_{j=0}^{n}(\varepsilon^j J_3^j(\bar{\theta}) + \varepsilon^{j+\frac{1}{2}}J_3^{j+\frac{1}{2}}(\bar{\theta})), \end{cases} \tag{3.40}$$

where

$$\begin{cases} J_1^j(\bar{\theta}) = \sum_{k=0}^{j-1}(\kappa(\eta)\bar{\xi})^{2(j-k)-1}\frac{\partial\bar{\theta}^k}{\partial\bar{\xi}} + \sum_{k=0}^{j-1}(\kappa(\eta)\bar{\xi})^{2(j-k)-2}\frac{\partial\bar{\theta}^{k+\frac{1}{2}}}{\partial\bar{\xi}}, \\ J_1^{j+\frac{1}{2}}(\bar{\theta}) = \sum_{k=0}^{j}(\kappa(\eta)\bar{\xi})^{2(j-k)}\frac{\partial\bar{\theta}^k}{\partial\bar{\xi}} + \sum_{k=0}^{j-1}(\kappa(\eta)\bar{\xi})^{2(j-k)-1}\frac{\partial\bar{\theta}^{k+\frac{1}{2}}}{\partial\bar{\xi}}, \end{cases} \tag{3.41}$$

$$\begin{cases} J_2^j(\bar{\theta}) = \sum_{k=0}^{j-1}(2(j-k)-1)(\kappa(\eta)\bar{\xi})^{2(j-k)-2}\frac{\partial^2\bar{\theta}^k}{\partial\eta^2} + \sum_{k=0}^{j-2}(2(j-k)-2)(\kappa(\eta)\bar{\xi})^{2(j-k)-3}\frac{\partial^2\bar{\theta}^{k+\frac{1}{2}}}{\partial\eta^2}, \\ J_2^{j+\frac{1}{2}}(\bar{\theta}) = \sum_{k=0}^{j-1}(2(j-k))(\kappa(\eta)\bar{\xi})^{2(j-k)-1}\frac{\partial^2\bar{\theta}^k}{\partial\eta^2} + \sum_{k=0}^{j-1}(2(j-k)-1)(\kappa(\eta)\bar{\xi})^{2(j-k)-2}\frac{\partial^2\bar{\theta}^{k+\frac{1}{2}}}{\partial\eta^2}, \end{cases} \tag{3.42}$$

$$\begin{cases} J_3^j(\bar{\theta}) = \sum_{k=0}^{j-2}(2(j-k)-2)(2(j-k)-1)(\kappa(\eta)\bar{\xi})^{2(j-k)-3}\frac{\partial\bar{\theta}^k}{\partial\eta} \\ \qquad + \sum_{k=0}^{j-2}(2(j-k)-3)(2(j-k)-2)(\kappa(\eta)\bar{\xi})^{2(j-k)-4}\frac{\partial\bar{\theta}^{k+\frac{1}{2}}}{\partial\eta}, \\ J_3^{j+\frac{1}{2}}(\bar{\theta}) = \sum_{k=0}^{j-1}(2(j-k)-1)(2(j-k))(\kappa(\eta)\bar{\xi})^{2(j-k)-2}\frac{\partial\bar{\theta}^k}{\partial\eta} \\ \qquad + \sum_{k=0}^{j-2}(2(j-k)-2)(2(j-k)-1)(\kappa(\eta)\bar{\xi})^{2(j-k)-3}\frac{\partial\bar{\theta}^{k+\frac{1}{2}}}{\partial\eta}. \end{cases} \tag{3.43}$$

We note that when the upper limit and exponents in the expressions of J_k^j and $J_k^{j+\frac{1}{2}}$ are negative these terms are set to be zero. To handle the nonlinear term g, we derive the analogue of Lemma 2.6. Replacing ε by $\sqrt{\varepsilon}$, n by $2n+d$ ($d = 0, 1$), and then renaming u^{2j}, θ^{2j}, u^{2j+1}, θ^{2j+1}, respectively, as u^j, θ^j, $u^{j+\frac{1}{2}}$, $\theta^{j+\frac{1}{2}}$ and setting $u^{j+\frac{1}{2}} = 0$, we obtain the following lemma.

Lemma 3.8. *There exists a constant $C > 0$, independent of ε, such that*

$$\begin{cases} \left|g\Big(\sum_{j=0}^{n}(\varepsilon^j u^j + \varepsilon^j\theta^j) + \sum_{j=0}^{n-1}\varepsilon^{j+\frac{1}{2}}\theta^{j+\frac{1}{2}}\Big) - g\Big(\sum_{j=0}^{n}\varepsilon^j u^j\Big) - G_n\right| \le C\sqrt{\varepsilon}^{2n+1}, \\ \left|g\Big(\sum_{j=0}^{n}(\varepsilon^j u^j + \varepsilon^j\theta^j + \varepsilon^{j+\frac{1}{2}}\theta^{j+\frac{1}{2}})\Big) - g\Big(\sum_{j=0}^{n}\varepsilon^j u^j\Big) - G_{n+\frac{1}{2}}\right| \le C\sqrt{\varepsilon}^{2n+2}, \end{cases} \tag{3.44}$$

where, for $d = 0, 1$,

$$G_{n+\frac{d}{2}} = \sum_{m=0}^{2n+d}\Big\{\sum_{k=0}^{m}\sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+m\alpha_m=m}}\binom{k}{\alpha}\frac{1}{k!}[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha - g^{(k)}(u^0)u^\alpha]\Big\}\sqrt{\varepsilon}^m, \tag{3.45}$$

and the multi-index notations are defined, for $d = 0, 1$, as follows

$$\begin{cases} \alpha = (\alpha_1, \dots, \alpha_{2j+d}), & |\alpha| = \alpha_1 + \cdots + \alpha_{2j+d}, \\ (u+\theta)^\alpha = (u^{\frac{1}{2}} + \theta^{\frac{1}{2}})^{\alpha_1} \dots (u^{j+\frac{d}{2}} + \theta^{j+\frac{d}{2}})^{\alpha_{2j+d}}, & u^\alpha = (u^{\frac{1}{2}})^{\alpha_1} \dots (u^{j+\frac{d}{2}})^{\alpha_{2j+d}}, \end{cases} \tag{3.46}$$

with $u^{j+\frac{1}{2}} = (u^{j+\frac{1}{2}})^{\alpha_{2j+1}} = 0$ *if* $\alpha_{2j+1} \geq 1$ *and* $(u^{j+\frac{1}{2}})^0 = 1$ *for* $j = 0, 1, 2, \dots$.

We now construct high order boundary layers. From the formal sum (3.37) and Lemma 3.8 with $n = \infty$, we can write

$$\begin{aligned} &-\varepsilon\Delta\Big(\sum_{j=0}^{\infty} \varepsilon^j(\theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}})\Big) \\ &\quad + \sum_{m=0}^{\infty}\Big\{\sum_{k=0}^{m} \sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+m\alpha_m=m}} \binom{k}{\alpha}\frac{1}{k!}\big[g^{(k)}(u^0+\theta^0)(u+\theta)^\alpha - g^{(k)}(u^0)u^\alpha\big]\Big\}\sqrt{\varepsilon}^m = 0. \end{aligned} \tag{3.47}$$

We then observe that at order $\mathcal{O}(\varepsilon)$

$$-\bar{\theta}^1_{\bar{\xi}\bar{\xi}} + g'(u^0+\bar{\theta}^0)\bar{\theta}^1 = (g'(u^0) - g'(u^0+\bar{\theta}^0))u^1 - \frac{g''(u^0+\bar{\theta}^0)}{2}(\bar{\theta}^{\frac{1}{2}})^2 - \kappa(\eta)\big((\kappa(\eta)\bar{\xi}\bar{\theta}^0_{\bar{\xi}} + \bar{\theta}_{\bar{\xi}}^{\frac{1}{2}}) + \bar{\theta}^0_{\eta\eta}.$$

Incorporating (3.39) and (3.40), and from (3.47) at $m = 2j$, we obtain at order $\mathcal{O}(\varepsilon^j)$ for $j \geq 1$ that

$$\begin{aligned} -\bar{\theta}^j_{\bar{\xi}\bar{\xi}} + g'(u^0+\bar{\theta}^0)\bar{\theta}j &= -(g'(u^0+\bar{\theta}^0) - g'(u^0))u^j \\ &\quad - \Big\{\sum_{k=2}^{2j} \sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+2j\alpha_{2j}=2j}} \binom{k}{\alpha}\frac{1}{k!}\big[g^{(k)}(u^0+\bar{\theta}^0)(u+\bar{\theta})^\alpha - g^{(k)}(u^0)u^\alpha\big]\Big\} \\ &\quad - \kappa(\eta)J_1^j(\bar{\theta}) + J_2^j(\bar{\theta}) + \frac{\bar{\xi}}{2}\kappa'(\eta)J_3^j(\bar{\theta}), \end{aligned} \tag{3.48}$$

where the multi-index notation is described in (3.46). On the other hand, at order $\mathcal{O}(\varepsilon^{j+\frac{1}{2}})$ for $j \geq 1$, from (3.47) at $m = 2j+1$ we similarly find that

$$\begin{aligned} -\bar{\theta}^{j+\frac{1}{2}}_{\bar{\xi}\bar{\xi}} + g'(u^0+\bar{\theta}^0)\bar{\theta}^{j+\frac{1}{2}} &= -(g'(u^0+\bar{\theta}^0) - g'(u^0))u^{j+\frac{1}{2}} \\ &\quad - \Big\{\sum_{k=2}^{2j+1} \sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+(2j+1)\alpha_{2j+1}=2j+1}} \binom{k}{\alpha}\frac{1}{k!}\big[g^{(k)}(u^0+\bar{\theta}^0)(u+\bar{\theta})^\alpha - g^{(k)}(u^0)u^\alpha\big]\Big\} \\ &\quad - \kappa(\eta)J_1^{j+\frac{1}{2}}(\bar{\theta}) + J_2^{j+\frac{1}{2}}(\bar{\theta}) + \frac{\bar{\xi}}{2}\kappa'(\eta)J_3^{j+\frac{1}{2}}(\bar{\theta}). \end{aligned} \tag{3.49}$$

The two equations (3.48) and (3.49) are, respectively, supplemented with the boundary conditions

$$\begin{cases} \bar{\theta}^j = -u^j|_{\xi=0} = -u^j(X(\eta), Y(\eta)) & \text{at } \xi = 0, \\ \bar{\theta}^j = 0 & \text{at } \xi = \xi_0, \end{cases} \qquad \begin{cases} \bar{\theta}^{j+\frac{1}{2}} = 0 & \text{at } \xi = 0, \\ \bar{\theta}^{j+\frac{1}{2}} = 0 & \text{at } \xi = \xi_0. \end{cases}$$

As before, we extend $\bar{\theta}^j$ and $\bar{\theta}^{j+\frac{1}{2}}$ by zero for $\xi > \xi_0$ and these extensions are still denoted by the same notations.

Multiplying, respectively, (3.14) by ε^0, (3.15) by $\varepsilon^{\frac{1}{2}}$, (3.48) by ε^j and (3.49) by $\varepsilon^{j+\frac{1}{2}}$, from $j = 1$ to $j = n$, and adding these resulting equations we find that

$$\begin{aligned} &-\sum_{j=0}^{n} \varepsilon^j(\bar{\theta}^j_{\bar{\xi}\bar{\xi}} + \sqrt{\varepsilon}\bar{\theta}^{j+\frac{1}{2}}_{\bar{\xi}\bar{\xi}}) \\ &\quad = -\sum_{m=0}^{2n+1}\Big\{\sum_{k=0}^{m} \sum_{\substack{|\alpha|=k,\\ \alpha_1+2\alpha_2+\cdots+m\alpha_m=m}} \binom{k}{\alpha}\frac{1}{k!}\big[g^{(k)}(u^0+\bar{\theta}^0)(u+\bar{\theta})^\alpha - g^{(k)}(u^0)u^\alpha\big]\Big\}\sqrt{\varepsilon}^m + I_1^n + I_2^n + I_3^n, \end{aligned} \tag{3.50}$$

where I_k^n are described as in (3.40).

Lemma 3.9. *For $l, m, n \geq 0$ and $j = 0, 1, 2, \ldots$, there exist $c > 0$ such that, pointwise,*

$$\left|\xi^n \frac{\partial^{l+m}\bar{\theta}^j}{\partial\xi^l\partial\eta^m}\right| + \left|\xi^n \frac{\partial^{l+m}\bar{\theta}^{j+\frac{1}{2}}}{\partial\xi^l\partial\eta^m}\right| \leq c\varepsilon^{\frac{n-l}{2}} \exp\Big(-\frac{1}{2}\sqrt{\frac{\lambda}{\varepsilon}}\xi\Big). \tag{3.51}$$

Moreover, $\bar{\theta}^j$ and $\bar{\theta}^{j+\frac{1}{2}}$ satisfy, for $j = 0, 1, 2, \ldots$,

$$\left\|\xi^n \frac{\partial^{l+m}\bar{\theta}^j}{\partial\xi^l\partial\eta^m}\right\|_{L^2_\xi(\Omega)} + \left\|\xi^n \frac{\partial^{l+m}\bar{\theta}^{j+\frac{1}{2}}}{\partial\xi^l\partial\eta^m}\right\|_{L^2_\xi(\Omega)} \leq c\varepsilon^{\frac{n-l}{2}+\frac{1}{4}}. \tag{3.52}$$

Proof. Using Lemma 3.3 and Lemma 3.4 and the induction on j, from (3.48) and (3.49), we obtain the estimates (3.19) and (3.25) for $\bar{\theta}^j$ and $\bar{\theta}^{j+\frac{1}{2}}$, $j = 0, 1, 2, \ldots$. Here, as indicated in (2.38), for $k \geq 2$ we used the fact that the right-hand side of (3.48) and (3.49) involve only the preceding boundary layer correctors. From the pointwise estimates (3.51), we readily obtain the L^2- norm estimates (3.52). □

Lemma 3.10. *There exists $c > 0$ such that*

$$\left\|\Big(\varepsilon\Delta\Big(\sum_{j=0}^{n}(\varepsilon^j\bar{\theta}^j + \varepsilon^{j+\frac{1}{2}}\bar{\theta}^{j+\frac{1}{2}})\Big) - I^n\Big)\tilde{\delta}(\xi)\right\|_{L^2(\Omega)} \leq c\varepsilon^{n+\frac{5}{4}},$$

where $\tilde{\delta}(\xi)$ is given in (3.28), *and*

$$I^n = \sum_{j=0}^{n} \varepsilon^j(\bar{\theta}^j + \sqrt{\varepsilon}\bar{\theta}^{j+\frac{1}{2}})_{\bar{\xi}\bar{\xi}} + I_1^n + I_2^n + I_3^n$$

with I_i^n given by (3.40).

Proof. Using the Laplacian (3.7) in terms of ξ, η, we write

$$\varepsilon\Delta\Big(\sum_{j=0}^{n}(\varepsilon^j\bar{\theta}^j + \varepsilon^{j+\frac{1}{2}}\bar{\theta}^{j+\frac{1}{2}})\Big) - \varepsilon\sum_{j=0}^{n}(\varepsilon^j\bar{\theta}^j + \varepsilon^{j+\frac{1}{2}}\bar{\theta}^{j+\frac{1}{2}})_{\xi\xi} = K_1^n + K_2^n + K_3^n, \tag{3.53}$$

where

$$K_1^n = -\varepsilon\kappa(\eta)\sigma(\xi,\eta)\frac{\partial}{\partial\xi}\Big(\sum_{j=0}^{n}(\varepsilon^j\bar{\theta}^j + \varepsilon^{j+\frac{1}{2}}\bar{\theta}^{j+\frac{1}{2}})\Big),$$

$$K_2^n = \varepsilon\sigma^2(\xi,\eta)\frac{\partial^2}{\partial\eta^2}\Big(\sum_{j=0}^{n}(\varepsilon^j\bar{\theta}^j + \varepsilon^{j+\frac{1}{2}}\bar{\theta}^{j+\frac{1}{2}})\Big),$$

$$K_3^n = \varepsilon\xi\kappa'(\eta)\sigma^3(\xi,\eta)\frac{\partial}{\partial\eta}\Big(\sum_{j=0}^{n}(\varepsilon^j\bar{\theta}^j + \varepsilon^{j+\frac{1}{2}}\bar{\theta}^{j+\frac{1}{2}})\Big).$$

We now only have to prove that

$$\|(K_i^n - I_i^n)\tilde{\delta}(\xi)\|_{L^2} \leq \kappa\varepsilon^{n+\frac{5}{4}}, \quad i = 1, 2, 3. \tag{3.54}$$

Noting that $\sum_{j=0}^{n}\sum_{k=0}^{j} = \sum_{k=0}^{n}\sum_{j=k}^{n}$ and $\sum_{j=0}^{n}\sum_{k=0}^{j-1} = \sum_{k=0}^{n}\sum_{j=k+1}^{n}$, from (3.40) and (3.41) we write

$$\begin{aligned}
I_1^n = {} & -\kappa(\eta)\sum_{k=0}^{n}\Big(\sum_{j=k+1}^{n}\varepsilon^j(\kappa(\eta)\bar{\xi})^{2(j-k)-1} + \sum_{j=k}^{n}\varepsilon^{j+\frac{1}{2}}(\kappa(\eta)\bar{\xi})^{2(j-k)}\Big)\frac{\partial\bar{\theta}^k}{\partial\bar{\xi}} \\
& -\kappa(\eta)\sum_{k=0}^{n}\Big(\sum_{j=k+1}^{n}\varepsilon^j(\kappa(\eta)\bar{\xi})^{2(j-k)-2} + \sum_{j=k+1}^{n}\varepsilon^{j+\frac{1}{2}}(\kappa(\eta)\bar{\xi})^{2(j-k)-1}\Big)\frac{\partial\bar{\theta}^{k+\frac{1}{2}}}{\partial\bar{\xi}} \\
= {} & -\kappa(\eta)\sum_{k=0}^{n}\Big(\varepsilon^{k+1}\sum_{m=0}^{2(n-k)}(\kappa(\eta)\xi)^m\Big)\frac{\partial\bar{\theta}^k}{\partial\xi} - \kappa(\eta)\sum_{k=0}^{n}\Big(\varepsilon^{k+\frac{3}{2}}\sum_{m=0}^{2(n-k)-1}(\kappa(\eta)\xi)^m\Big)\frac{\partial\bar{\theta}^{k+\frac{1}{2}}}{\partial\xi}.
\end{aligned}$$

Then, we find

$$\begin{aligned}\|(K_1^n - I_1^n)\tilde{\delta}(\xi)\|_{L^2} &\le \Big\|\varepsilon\kappa(\eta)\Big(\sum_{j=0}^{n}\varepsilon^j\Big(\sigma(\xi,\eta) - \sum_{m=0}^{2(n-j)}(\kappa(\eta)\xi)^m\Big)\frac{\partial\bar{\theta}^j}{\partial\xi}\Big)\tilde{\delta}(\xi)\Big\|_{L^2}\\ &\quad + \Big\|\varepsilon\kappa(\eta)\Big(\sum_{j=0}^{n}\varepsilon^{j+\frac{1}{2}}\Big(\sigma(\xi,\eta) - \sum_{m=0}^{2(n-j)-1}(\kappa(\eta)\xi)^m\Big)\frac{\partial\bar{\theta}^{j+\frac{1}{2}}}{\partial\xi}\Big)\tilde{\delta}(\xi)\Big\|_{L^2}\\ &\le c\varepsilon\sum_{j=0}^{n}\Big(\Big\|\varepsilon^j(\kappa(\eta)\xi)^{2(n-j)+1}\tilde{\delta}(\xi)\frac{\partial\bar{\theta}^j}{\partial\xi}\Big\|_{L^2} + \Big\|\varepsilon^{j+\frac{1}{2}}(\kappa(\eta)\xi)^{2(n-j)}\tilde{\delta}(\xi)\frac{\partial\bar{\theta}^{j+\frac{1}{2}}}{\partial\xi}\Big\|_{L^2}\Big)\\ &\le c\varepsilon^{n+\frac{5}{4}},\end{aligned}$$

where in the last inequality we used Lemma 3.9. Note that in (3.42) and (3.43) the sum $\sum_{k=0}^{j-2}$ can be replaced by $\sum_{k=0}^{j-1}$ because the terms there for $k = j-1$ do not contribute. Then, permuting summations as above, we can also prove (3.54) for $i = 2, 3$. The lemma thus follows. □

We use Lemmas 3.8 and 3.10, and equation (3.50), to find that

$$-\varepsilon\Delta\Big(\sum_{j=0}^{n}(\varepsilon^j\bar{\theta}^j + \varepsilon^{j+\frac{1}{2}}\bar{\theta}^{j+\frac{1}{2}})\Big) + g\Big(\sum_{j=0}^{n}(\varepsilon^j u^j + \varepsilon^j\bar{\theta}^j + \varepsilon^{j+\frac{1}{2}}\bar{\theta}^{j+\frac{1}{2}})\Big) - g\Big(\sum_{j=0}^{n}\varepsilon^j u^j\Big) = R^n + R, \tag{3.55}$$

where

$$\|R^n\tilde{\delta}(\xi)\|_{L^2} \le c\varepsilon^{n+\frac{5}{4}}, \quad |R| \le c\sqrt{\varepsilon}^{2n+2}.$$

As before, we define now $\theta^j = \bar{\theta}^j\delta(\xi)$ and $\theta^{j+\frac{1}{2}} = \bar{\theta}^{j+\frac{1}{2}}\delta(\xi)$ for each $j \ge 0$, where δ is defined as in (3.17). Note that θ^j, $\theta^{j+\frac{1}{2}}$ satisfy the boundary conditions as in (3.12).

Theorem 3.11. *Assume that f is a smooth function, Ω is a general smooth domain and u^ε is solution of* (2.1)*. Let u^j and θ^j satisfy* (3.3) *and* (3.48)*, respectively. Then, for every $n \ge 0$, there exists a constant $c > 0$, independent of ε, such that*

$$\Big\|u^\varepsilon - \sum_{j=0}^{n}\varepsilon^j(u^j + \theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}})\Big\|_\varepsilon \le c\varepsilon^{n+1}, \tag{3.56}$$

$$\Big\|u^\varepsilon - \sum_{j=0}^{n}\varepsilon^j(u^j + \theta^j) - \sum_{j=0}^{n-1}\varepsilon^{j+\frac{1}{2}}\theta^{j+\frac{1}{2}}\Big\|_\varepsilon \le c\varepsilon^{n+\frac{3}{4}}. \tag{3.57}$$

Proof. We set

$$w^{n+\frac{1}{2}} = u^\varepsilon - \sum_{j=0}^{n}\varepsilon^j(u^j + \theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}}).$$

We use the smooth cut-off function $\tilde{\delta}(\xi)$ to eliminate the singularity of $\sigma(\xi,\eta)$ where $\tilde{\delta}(\xi)$ is given by (3.28). Then, by a similar argument as before, we obtain that

$$\begin{aligned}&\Big(\varepsilon\Delta\sum_{j=0}^{n}\varepsilon^j(\theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}}) - g\Big(\sum_{j=0}^{n}\varepsilon^j(u^j + \theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}})\Big) + g\Big(\sum_{j=0}^{n}\varepsilon^j u^j\Big), w^{n+\frac{1}{2}}\Big)\\ &\quad = ((G_{n+\frac{1}{2}} + E_{n+\frac{1}{2}})\tilde{\delta}(\xi), w^{n+\frac{1}{2}}) + (R_{n+\frac{1}{2}}\tilde{\delta}(\xi), w^{n+\frac{1}{2}}),\end{aligned}$$

where

$$\begin{aligned}E_{n+\frac{1}{2}} &= \varepsilon\Delta\Big(\sum_{j=0}^{n}\varepsilon^j(\theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}}) - \sum_{j=0}^{n}\varepsilon^j(\bar{\theta}^j + \sqrt{\varepsilon}\bar{\theta}^{j+\frac{1}{2}})\Big),\\ G_{n+\frac{1}{2}} &= g\Big(\sum_{j=0}^{n}\varepsilon^j(u^j + \bar{\theta}^j + \sqrt{\varepsilon}\bar{\theta}^{j+\frac{1}{2}})) - g\Big(\sum_{j=0}^{n}\varepsilon^j(u^j + \theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}})),\\ R_{n+\frac{1}{2}} &= R^n + R,\end{aligned}$$

where R^n and R are given in (3.55). We note that $E_{n+\frac{1}{2}}$ and $G_{n+\frac{1}{2}}$ are e.s.t., and $\|R_{n+\frac{1}{2}}\tilde{\delta}(\xi)\|_{L^2(\Omega)} \le \varepsilon^{n+1}$. We now find

$$\Big(-\varepsilon\Delta w^{n+\frac{1}{2}} + g(u^\varepsilon) - g\Big(\sum_{j=0}^{n} \varepsilon^j(u^j + \theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}})\Big), w^{n+\frac{1}{2}}\Big)$$
$$= (\varepsilon^{n+1}\Delta u^n, w^{n+\frac{1}{2}}) + ((G_{n+\frac{1}{2}} + E_{n+\frac{1}{2}})\tilde{\delta}(\xi), w^{n+\frac{1}{2}}) + (R_{n+\frac{1}{2}}\tilde{\delta}(\xi), w^{n+\frac{1}{2}}).$$

Here, we used (2.41) which is obtained from summing (3.3).

We finally obtain from the mean value theorem that, for some ζ between u^ε and $\sum_{j=0}^{n} \varepsilon^j(u^j + \theta^j + \sqrt{\varepsilon}\theta^{j+\frac{1}{2}})$,

$$(-\varepsilon\Delta w^{n+\frac{1}{2}} + g'(\zeta)w^{n+\frac{1}{2}}, w^{n+\frac{1}{2}}) \le c\varepsilon^{n+1}\|w^{n+\frac{1}{2}}\|_{L^2(\Omega)}.$$

This proves (3.56). From Lemma 3.9, we note that $\|\varepsilon^{n+\frac{1}{2}}\theta^{n+\frac{1}{2}}\|_\varepsilon \le c\varepsilon^{n+\frac{3}{4}}$, and the estimate (3.57) follows from (3.56). □

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