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# Notes on the knot concordance invariant Upsilon

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Ozsváth, Stipsicz and Szabó have defined a knot concordance invariant $\Upsilon_K$ taking values in the group of piecewise linear functions on the closed interval $[0,2]$. This paper presents a description of one approach to defining $\Upsilon_K$ and proving its basic properties.

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## 1 Introduction

Ozsváth, Stipsicz and Szabó [5] used the Heegaard Floer knot complex $\mathrm{CFK}^-(K)$ of a knot $K \subset S^3$ to define a piecewise linear function $\Upsilon_K(t)$ with domain $[0,2]$. The function $K \to \Upsilon_K$ induces a homomorphism from the smooth knot concordance group to the group of functions on the interval $[0,2]$. Among its properties, $\Upsilon_K(t)$ provides bounds on the four-genus, $g_4(K)$, the three-genus, $g_3(K)$, and, consequently, the concordance genus, $g_c(K)$. This note describes a simple approach to defining $\Upsilon_K(t)$ using $\mathrm{CFK}^\infty(K)$ and proving its basic properties.

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## 2 Knot complexes

We begin by describing the algebraic structure of the Heegaard Floer complex of a knot $K$, denoted $\mathrm{CFK}^\infty(K)$, first defined in Ozsváth and Szabó [9]. This is a vector space over the field $\mathbb{F}$ with two elements. To simplify notation, we write $\mathrm{CF}(K)$ for $\mathrm{CFK}^\infty(K)$. Here we summarize its basic properties:

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- The chain complex $\mathrm{CF}(K)$ has an integer valued grading and the boundary map $\partial$ is of degree $-1$. The grading is called the *Maslov grading*. The grading of a homogeneous element is denoted $\mathrm{gr}(x)$.
- The complex $\mathrm{CF}(K)$ has an *Alexander filtration* consisting of an increasing sequence of subcomplexes. The filtration level of an element $x \in \mathrm{CF}(K)$ is denoted $\mathrm{Alex}(x)$.
- There is a similar filtration, called the *algebraic filtration*, and filtration levels of elements are denoted $\mathrm{Alg}(x)$.
- There is an action of the Laurent polynomial ring $\mathbb{F}[U, U^{-1}]$ on $\mathrm{CF}(K)$. The action of $U$ commutes with $\partial$, lowers gradings by $2$, and lowers Alexander and algebraic filtration levels by $1$.
- Let $\Lambda$ denote $\mathbb{F}[U, U^{-1}]$. As a $\Lambda$–module, $\mathrm{CF}(K)$ is free on a finite set of generators, $\{x_i\}_{1 \le i \le r}$. To simplify notation, we suppress the indexing set. The set of elements $\{U^k x_i\}_{k \in \mathbb{Z}}$ forms a bifiltered graded basis for $\mathrm{CF}(K)$: for any triple of integers, $(g, m, n)$, the subspace of $\mathrm{CF}(K)$ spanned by elements of grading $g$, Alexander filtration level less than or equal to $m$, and algebraic filtration level less than or equal to $n$, has as basis a subset of $\{U^k x_i\}$.
- The singly filtered complex $(\mathrm{CF}(K), \mathrm{Alg})$ with $\Lambda$–structure is chain homotopy equivalent to complex $\mathcal{T} \cong \Lambda$ where $1 \in \Lambda$ has grading $0$ and filtration level $0$, and the boundary map is trivial. (The same statement holds for the Alexander grading, but we do not use this fact.)

The construction of $\mathrm{CF}(K)$ depends on a series of choices. However, there is a natural definition of chain homotopy equivalence for graded, bifiltered chain complexes with $\Lambda$–action. A key result of [9] is that in this sense, the chain homotopy equivalence class of $\mathrm{CF}(K)$ is a well-defined knot invariant.

As an example, Figure 1 presents a schematic diagram of the complex for the torus knot $T(3,7)$. As a $\Lambda$–module it has nine filtered generators, with algebraic and Alexander filtration levels indicated by the first and second coordinate, respectively. Five of the generators, indicated with black dots, have grading $0$; the four white dots represent generators of grading one. The boundary map is indicated by the arrows. The rest of $\mathrm{CF}(K)$ is the direct sum of the $U^k$ translates for $k \in \mathbb{Z}$ of this finite complex; for instance, applying $U$ shifts the diagram one down and to the left.

# 3 Filtrations

We now discuss more general filtrations on vector spaces. In our applications, the vector space will be $\mathrm{CF}(K)$.

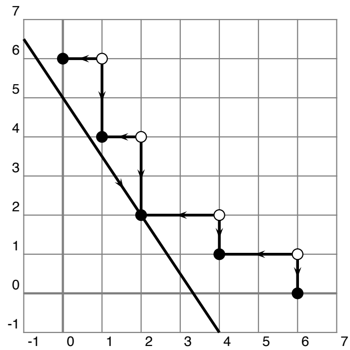


Figure 1: $\mathrm{CFK}^\infty(T(3,7))$

**Definition 3.1** A *real-valued (discrete) filtration* on a vector space C is a collection of subspaces $\mathcal{F} = \{\mathrm{C}_s\}$ indexed by the real numbers. This collection must satisfy the following properties:

(1) $\mathrm{C}_{s_1} \subseteq \mathrm{C}_{s_2}$ if $s_1 \leq s_2$.

(2) $\mathrm{C} = \bigcup_{s\in\mathbb{R}} \mathrm{C}_s$.

(3) $\bigcap_{s\in\mathbb{R}} \mathrm{C}_s = \{0\}$.

(4) **discreteness** $\mathrm{C}_{s_2}/\mathrm{C}_{s_1}$ is finite-dimensional when $s_1 \leq s_2$.

Given a discrete filtration $\mathcal{F} = \{\mathrm{C}_s\}$ on C, we can define an associated function on C, which we temporarily also denote by $\mathcal{F}$, given by $\mathcal{F}(x) = \min\{s \in \mathbb{R} \mid x \in \mathrm{C}_s\}$. Notice that $\mathcal{F}^{-1}((-\infty, s]) = \mathrm{C}_s$.

Given an arbitrary real-valued function $f$ on C, one can define an associated filtration with $\mathrm{C}_s = \mathrm{Span}(f^{-1}((-\infty, s]))$. The resulting filtration need not be discrete.

**Notation** In cases in which more than one filtration might be under consideration, we will write $(\mathrm{C}, \mathcal{F})_s$ rather than $\mathrm{C}_s$.

**Definition 3.2** A set of vectors $\{z_i\}$ in the real filtered vector space C is called a *filtered basis* if it is linearly independent and every $\mathrm{C}_s$ has some subset of $\{z_i\}$ as a basis. If C is also graded, $\mathrm{C} = \bigoplus_{i=-\infty}^{\infty} G_i$, then we say the basis is a filtered graded basis if each $\mathrm{C}_s \cap G_k$ has a subset of $\{z_i\}$ as a basis.

# 4 The definition of the filtration $\mathcal{F}_t$ on $\mathrm{CF}(K)$

For any $t \in [0,2]$, the convex combination of Alexander and algebraic filtrations, $\frac{t}{2}\,\mathrm{Alex} + \left(1-\frac{t}{2}\right)\mathrm{Alg}$, defines a real-valued function on $\mathrm{CF}(K)$, to which we associate a filtration denoted $\mathcal{F}_t$. That is, for all $s \in \mathbb{R}$, $(\mathrm{CF}(K), \mathcal{F}_t)_s$ is spanned by all vectors $x \in \mathrm{CF}(K)$ such that $\frac{t}{2}\,\mathrm{Alex}(x) + \left(1-\frac{t}{2}\right)\mathrm{Alg}(x) \le s$.

**Theorem 4.1** *If $0 \le t \le 2$, the filtration $\mathcal{F}_t$ on $\mathrm{CF}(K)$ is a filtration by subcomplexes and is discrete. The action of $U$ lowers filtration levels by $1$.*

**Proof** To see that these are subcomplexes, suppose that $x \in (\mathrm{CF}(K), \mathcal{F}_t)_s$. Write $x = \sum x_i$, where $\frac{t}{2}\,\mathrm{Alex}(x_i) + \left(1-\frac{t}{2}\right)\mathrm{Alg}(x_i) \le s$ for all $i$. Since $\partial x = \sum \partial x_i$, we only need to check that $\partial x_i \in (\mathrm{CF}(K), \mathcal{F}_t)_s$ for each $i$. Let $x_i$ have $\mathrm{Alex}(x_i) = a$ and $\mathrm{Alg}(x_i) = b$. Then $\mathrm{Alex}(\partial x_i) = a' \le a$ and $\mathrm{Alg}(\partial x_i) = b' \le b$. Since both $\frac{t}{2}$ and $1-\frac{t}{2}$ are nonnegative, $\frac{t}{2}a' + \left(1-\frac{t}{2}\right)b' \le \frac{t}{2}a + \left(1-\frac{t}{2}\right)b \le s$, as desired.

The discreteness of the filtration depends on two properties of $\mathrm{CF}(K)$. First, letting $g$ denote the three-genus, $g_3(K)$, according to [8] one has $-g \le \mathrm{Alex}(x) - \mathrm{Alg}(x) \le g$ for all $x$. From this it follows that for given $s_1 < s_2$, there are $k_1$ and $k_2$ in $\mathbb{R}$ such that

$$(\mathrm{CF}(K), \mathrm{Alex})_{k_1} \subseteq (\mathrm{CF}(K), \mathcal{F}_t)_{s_1} \subseteq (\mathrm{CF}(K), \mathcal{F}_t)_{s_2} \subseteq (\mathrm{CF}(K), \mathrm{Alex})_{k_2}.$$

(The values of $k_1$ and $k_2$ can be chosen to be $s_1 - \left(1-\frac{t}{2}\right)g$ and $s_2 + \left(1-\frac{t}{2}\right)g$, respectively, but we do not need this level of detail.) Second, the Alexander filtration is discrete, so the quotient $(\mathrm{CF}(K), \mathrm{Alex})_{k_2}/(\mathrm{CF}(K), \mathrm{Alex})_{k_1}$ is finite-dimensional.

Finally, that $U$ lowers filtration levels by one is immediate. □

# 5 The definition of $\Upsilon_K(t)$

For each $t \in [0,2]$ and for all $s \in \mathbb{R}$, the set $(\mathrm{CF}(K), \mathcal{F}_t)_s \subset \mathrm{CF}(K)$ is a subcomplex. Thus, we can make the following definition:

**Definition 5.1** Let

$$\nu(\mathrm{CF}(K), \mathcal{F}_t) = \min\left\{s \mid H_0((\mathrm{CF}(K), \mathcal{F}_t)_s) \to H_0(\mathrm{CF}(K)) \text{ is surjective}\right\}.$$

**Definition 5.2** $\Upsilon_K(t) = -2\nu(\mathrm{CF}(K), \mathcal{F}_t)$.

### 5.1 Example

Consider the knot $K = T(3,7)$ with $\mathrm{CF}(K)$ as illustrated in Figure 1. The portion of the complex shown has homology $\mathbb{F}$ at grading $0$.

The subcomplex $(\mathrm{CF}(K), \mathcal{F}_t)_s$ is generated by the bifiltered generators with Alexander and algebraic filtration levels satisfying

$$\text{Alex} \leq \frac{2}{t}s + \left(1 - \frac{2}{t}\right) \text{Alg}\,. \tag{5-1}$$

**Observation** The lattice points which contain a filtered generator at filtration level $t$ all lie on a line of slope

$$m = 1 - \frac{2}{t},$$

with lattice points parametrized by the pair $(\text{Alg}, \text{Alex})$. Alternatively, if a line of slope $m$ contains distinct lattice points representing bifiltration levels of generators at the same $\mathcal{F}_t$ filtration level, then

$$t = \frac{2}{1-m}.$$

In the diagram for $T(3,7)$ shown in Figure 1, the illustrated line in the plane corresponds to $t = \frac{4}{5}$ and $s = 2$. Since the lower half-plane bounded by this line contains a generator of $H_0(\mathrm{CF}(K))$, while no half-plane bounded by a parallel line with smaller value of $s$ contains such a generator, we have $\Upsilon_K\left(\frac{4}{5}\right) = -2(2) = -4$.

Continuing with $K = T(3,7)$, it is now clear that for $m < -2$ — that is, for $t < \frac{2}{3}$ — the least $s$ for which $(\mathrm{CF}(K), \mathcal{F}_t)_s$ contains a generator of $H_0(\mathrm{CF}(K))$ corresponds to the line through $(0,6)$, which has filtration level $\frac{t}{2}6 + \left(1 - \frac{t}{2}\right)0 = 3t$.

For $-2 < m < -1$ — that is, for $\frac{2}{3} < t < 1$ — the least $s$ for which $(\mathrm{CF}(K), \mathcal{F}_t)_s$ contains a generator of $H_0(\mathrm{CF}(K))$ corresponds to the line through $(2,2)$, which has filtration level $\frac{t}{2}2 + \left(1 - \frac{t}{2}\right)2 = 2$. Multiplying by $-2$ and checking the value $t = \frac{2}{3}$ yields

$$\Upsilon_{T(3,7)}(t) = \begin{cases} -6t & \text{if } 0 \leq t \leq \frac{2}{3}, \\ -4 & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

## 6 An alternative definition of $\nu$ and $\Upsilon$

In the appendix we prove Theorem A.1, which has as an immediate consequence the following result:

**Theorem 6.1** *The filtered graded chain complex* $(\mathrm{CF}(K), \mathcal{F}_t)$ *is isomorphic to a filtered graded complex of the form*

$$\mathcal{T} \oplus \mathcal{A},$$

*where* $\mathcal{T} \oplus \mathcal{A}$ *has the structure of a* $\Lambda$*–module and the isomorphism is a* $\Lambda$*–module isomorphism. The summand* $\mathcal{T}$ *has the properties that*

(1) *it is isomorphic to* $\Lambda$ *as a* $\Lambda$*–module;*

(2) *the element* $1 \in \Lambda \cong \mathcal{T}$ *has grading* $0$*.*

*Furthermore,* $\mathcal{A}$ *is acyclic as an unfiltered complex.*

Notice that since all gradings in $\mathcal{T}$ are even, the boundary operator restricted to $\mathcal{T}$ is trivial.

When placed in this simple form, the computation of $\nu((\mathrm{CF}(K), \mathcal{F}_t))$ is simple: it is the $\mathcal{F}_t$ filtration level of $1 \in \Lambda \cong \mathcal{T}$. Hence, we have the following result:

**Corollary 6.2** $\Upsilon_K(t)$ *equals* $-2$ *times the* $\mathcal{F}_t$*–filtration level of* $1 \in \Lambda \cong \mathcal{T}$ *for the decomposition* $(\mathrm{CF}(K), \mathcal{F}_t) \cong \mathcal{T} \oplus \mathcal{A}$*.*

# 7 Products and additivity

According to [9], there is a (graded) chain homotopy equivalence of complexes

$$\mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2) \simeq \mathrm{CF}(K_1 \# K_2)$$

that preserves the $\Lambda$–structure.

Each of $\mathrm{CF}(K_1)$, $\mathrm{CF}(K_2)$ and $\mathrm{CF}(K_1 \# K_2)$ has an algebraic filtration. To distinguish these, we write $\mathrm{Alg}^1$, $\mathrm{Alg}^2$ and $\mathrm{Alg}^{1,2}$. Similarly, the Alexander and $\mathcal{F}_t$ filtrations will be distinguished with superscripts.

Momentarily we write $\mathrm{CF}_1 = \mathrm{CF}(K_1)$ and $\mathrm{CF}_2 = \mathrm{CF}(K_2)$. For each $t \in [0,2]$ the filtrations $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ on $\mathrm{CF}_1$ and $\mathrm{CF}_2$ induce a filtration $\mathcal{F}_t^1 \otimes \mathcal{F}_t^2$ on $\mathrm{CF}_1 \otimes_\Lambda \mathrm{CF}_2$, defined via

$$\begin{aligned}(\mathrm{CF}_1 &\otimes_\Lambda \mathrm{CF}_2, \mathcal{F}_t^1 \otimes \mathcal{F}_t^2)_s \\ &= \mathrm{Image}\Big( \bigoplus_{s_1+s_2=s} (\mathrm{CF}_1, \mathcal{F}_t^1)_{s_1} \otimes_{\mathbb{F}} (\mathrm{CF}_2, \mathcal{F}_t^2)_{s_2} \to (\mathrm{CF}_1, \mathcal{F}_t^1) \otimes_\Lambda (\mathrm{CF}_2, \mathcal{F}_t^2)\Big).\end{aligned}$$

Notice that the direct sum is infinite and each summand is infinitely generated. Again, according to [9], for the connected sum of knots, the equivalence

$$\mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2) \simeq \mathrm{CF}(K_1 \# K_2)$$

is a filtered equivalence for both the Alexander and algebraic filtrations. To state this explicitly,

$$(\mathrm{CF}(K_1), \mathrm{Alex}^1) \otimes_\Lambda (\mathrm{CF}(K_2), \mathrm{Alex}^2) \simeq (\mathrm{CF}(K_1 \# K_2), \mathrm{Alex}^{1,2})$$

and

$$(\mathrm{CF}(K_1), \mathrm{Alg}^1) \otimes_\Lambda (\mathrm{CF}(K_2), \mathrm{Alg}^2) \simeq (\mathrm{CF}(K_1 \# K_2), \mathrm{Alg}^{1,2}).$$

**Theorem 7.1** *For all $t \in [0,1]$,*

$$(\mathrm{CF}(K_1), \mathcal{F}_t^1) \otimes_\Lambda (\mathrm{CF}(K_2), \mathcal{F}_t^2) \simeq (\mathrm{CF}(K_1 \# K_2), \mathcal{F}_t^{1,2}).$$

**Proof** Fix bases $\{x_i\}$ and $\{y_i\}$ for the free $\Lambda$–modules $\mathrm{CF}(K_1)$ and $\mathrm{CF}(K_2)$ such that the sets of all translates $\{U^k x_i\}$ and $\{U^k y_i\}$ for $k \in \mathbb{Z}$ form graded bifiltered bases for $\mathrm{CF}(K_1)$ and $\mathrm{CF}(K_2)$ (as $\mathbb{F}$–vector spaces). The $\mathbb{F}$–vector space $\mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$ is generated by the set of all tensor products, $\{U^k x_i \otimes U^j x_l\}$, but note that these do not form a basis; for instance, $Ux \otimes y = x \otimes Uy$.

When selecting elements from $\{U^k x_i\}$, we will sometimes refer to them as $x$, and similarly for $y$. Note that in particular, for such basis elements, $\mathrm{Alg}^{1,2}(x \otimes y) = \mathrm{Alg}^1(x) + \mathrm{Alg}^2(y)$ and $\mathrm{Alex}^{1,2}(x \otimes y) = \mathrm{Alex}^1(x) + \mathrm{Alex}^2(y)$.

The proof of the theorem consists of showing that the filtrations $\mathcal{F}_t^1 \otimes \mathcal{F}_t^2$ and $\mathcal{F}_t^{1,2}$ on $\mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$ are the same.

If an element $z \in \mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$ has $\mathcal{F}_t^{1,2}$ filtration level $s$, then it can be written as the sum of elements $x \otimes y$ with

$$\frac{t}{2}\,\mathrm{Alex}(x \otimes y) + \left(1 - \frac{t}{2}\right)\mathrm{Alg}(x \otimes y) \le s.$$

This is the same as

$$\frac{t}{2}\,\mathrm{Alex}(x) + \left(1 - \frac{t}{2}\right)\mathrm{Alg}(x) + \frac{t}{2}\,\mathrm{Alex}(y) + \left(1 - \frac{t}{2}\right)\mathrm{Alg}(y) \le s.$$

This implies that $\mathcal{F}_t^1(x) + \mathcal{F}_t^2(y) \le s$. This in turn implies that $(\mathcal{F}_t^1 \otimes \mathcal{F}_t^2)(x \otimes y) \le s$. Thus, $(\mathcal{F}_t^1 \otimes \mathcal{F}_t^2)(z) \le \mathcal{F}_t^{1,2}(z)$ for all $z \in \mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$.

Similarly, suppose that $z \in \mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$ has $\mathcal{F}_t^1 \otimes \mathcal{F}_t^2$ filtration level $s$. Then it is the sum of elements $x \otimes y$, each of which satisfies $\mathcal{F}_1^t(x) + \mathcal{F}_2^t(y) \le s$. This can be expanded and rewritten as

$$\frac{t}{2}(\mathrm{Alex}(x) + \mathrm{Alex}(y)) + \left(1 - \frac{t}{2}\right)(\mathrm{Alg}(x) + \mathrm{Alg}(y)) \le s.$$

In other words, $z$ is the sum of elements $x \otimes y$ with $\mathcal{F}_t^{1,2}(x \otimes y) \leq s$. Hence, $\mathcal{F}_t^{1,2}(x \otimes y) \leq s$. □

Theorem 7.1, along with Theorem 6.1, offers a fast proof of the additivity of $\Upsilon$:

**Theorem 7.2** *$\Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t)$ for each $t \in [0,2]$.*

**Proof** One only needs to check this for complexes of the form $\mathcal{T} \oplus \mathcal{A}$, as given in Theorem 6.1. Acyclic summands do not affect the value of $\Upsilon_K(t)$. Thus, we only need consider the case of complexes $\mathcal{T}(K_1) \otimes_\Lambda \mathcal{T}(K_2)$, for which the statement is clear. □

Similarly, Theorem 6.1 offers a fast proof of the following:

**Theorem 7.3** *$\Upsilon_{-K}(t) = -\Upsilon_K(t)$ for an arbitrary knot $K$.*

**Proof** According to [9], the complexes $\mathrm{CF}(K)$ and $\mathrm{CF}(-K)$ are duals: $\mathrm{CF}(-K) \cong \mathrm{CF}(K)^*$. More precisely, $\mathrm{CF}(-K)$ is isomorphic to the complex $\mathrm{Hom}_{\mathbb{F}}(\mathrm{CF}(K), \mathbb{F})$, having underlying vector space the space of $\mathbb{F}$–homomorphisms with finite-dimensional (that is, finite) support.

If we fix a basis $\{x_i\}$ of $\mathrm{CF}(K)$ as a $\Lambda$–module such that the set $\{U^k x_i\}$ forms a graded bifiltered basis of $\mathrm{CF}(K)$, then we can denote the elements of the dual basis by $(U^k x_i)^*$. The dual complex is readily understood in terms of these bases:

(1) An easy exercise shows that the action of $U$ on the dual basis is of the form $U(U^k x_i)^* = (U^{k-1} x_i)^*$. In particular, the set $\{x_i^*\}$ forms a basis for the $\Lambda$–module $\mathrm{CF}(K)^*$.

(2) For any filtration $\mathcal{F}$ on $\mathrm{CF}(K)$, we can define a filtration $\mathcal{F}^*$ on the dual space as follows:

$$(\mathrm{CF}(K)^*, \mathcal{F}^*)_s = \big\{\phi \in \mathrm{CF}(K)^* \mid \phi\big((\mathrm{CF}(K), \mathcal{F})_{-s'}\big) = 0 \text{ for all } s' > s\big\}.$$

The choice of signs ensures that the dual filtration is increasing. Thus, $\mathcal{F}^*(x_i^*) = -\mathcal{F}(x_i)$.

(3) The boundary operator for the dual space acts in the expected way with respect to basis elements: if $x$ is a component of $\partial y$, then $y^*$ is a component of $\partial x^*$.

These three observations are easily summarized in terms of diagrams such as in Figure 1: the diagram for $\mathrm{CF}(-K)$ is obtained from that for $\mathrm{CF}(K)$ by rotating the figure by 180 degrees around the origin and reversing all the arrows.

There are two filtrations on $\mathrm{CF}(-K)$ of interest. The first is $\frac{t}{2}\,\mathrm{Alex}^* + \left(1-\frac{t}{2}\right)\mathrm{Alg}^*$; the second is $\mathcal{F}_t^* = \left(\frac{t}{2}\,\mathrm{Alex} + \left(1-\frac{t}{2}\right)\mathrm{Alg}\right)^*$. By using the chosen basis and its dual basis, it is possible to see that these two filtrations are the same, as follows. We use coordinates $(i,j)$ for the plane. For a basis vector $x$, its dual vector $x^*$ is in $\mathcal{F}_t^*$ if and only if it lies on or above the line $\frac{t}{2}j + \left(1-\frac{t}{2}\right)i = -t$. If this is the case, then when rotated $180$ degrees about the origin it lies on or below the line $\frac{t}{2}j + \left(1-\frac{t}{2}\right)i = t$. These are precisely the dual vectors for which $\frac{t}{2}\,\mathrm{Alex}^* + \left(1-\frac{t}{2}\right)\mathrm{Alg}^* \le t$.

The proof of the theorem is now reduced to an elementary calculation for the simple complex $\mathcal{T}(K)$ and its dual $\mathcal{T}(K)^*$. □

# 8 Basic properties of $\Upsilon_K(t)$ and $\Upsilon'_K(t)$

We now present some basic results concerning $\Upsilon_K(t)$ and its derivative. An initial observation is that $\Upsilon_K(0) = 0$ and, since $\mathrm{CF}(K)$ is finitely generated, $\Upsilon_K(t)$ is continuous at $0$. Thus, we focus on $t > 0$.

**Theorem 8.1**

(1) *$\Upsilon(K)$ is a continuous piecewise linear function for every knot $K$.*

(2) *At a nonsingular point of $\Upsilon'_K(t)$, the value of $|\Upsilon'_K(t)|$ is $|i-j|$, where $(i,j)$ is the bifiltration level of some filtered generator of* $\mathrm{CF}(K)$ *with homological grading $0$.*

(3) *Singularities in $\Upsilon'_K(t)$ can occur only at values of $t$ such that some line of slope $1-\frac{2}{t}$ contains at least two lattice points, $(i,j)$ and $(i',j')$, each of which represents the algebraic and Alexander gradings of filtered generators of* $\mathrm{CF}(K)$ *of homological grading $0$.*

(4) *If $\Upsilon'_K(t)$ has a singularity at $t$, then the jump in $\Upsilon'_K(t)$ at $t$, denoted $\Delta\Upsilon'_K(t)$, satisfies $|\Delta\Upsilon'_K(t)| = \frac{2}{t}|i-i'|$ for some pair $(i,i')$ for which there are lattice points $(i,j)$ and $(i',j')$ as in the previous item.*

**Proof** The proof is discussed in terms of the diagram of the complex, as illustrated for the knot $T(3,7)$ in the previous section.

Suppose $\Upsilon_K(t) = -2s$ and there is exactly one lattice point $(i,j)$ with $\frac{t}{2}j + \left(1-\frac{t}{2}\right)i = s$ which represents the bifiltration level of a filtered generator of $\mathrm{CF}(K)$. (This will be the case for all but a finite number of values of $t$.) For a nearby $t$, say $t'$, the value of $\Upsilon_K(t') = -2s'$ will be such that the same vertex (at $(i,j)$) lies on the line

$\frac{t'}{2}j + \left(1 - \frac{t'}{2}\right)i = s'$. That is, for all nearby values of $t$, the value of $s$ is given by $\frac{t}{2}j + \left(1 - \frac{t}{2}\right)i$. Written differently,

$$\Upsilon_K(t) = -2i + (i - j)t.$$

In particular, we see that $\Upsilon_K(t)$ is piecewise linear off a finite set.

Now consider a singular value of $t$, at which $\Upsilon_K(t) = -2s$ and there are two or more pairs $(i, j)$ for which $\frac{t}{2}j + \left(1 - \frac{t}{2}\right)i = s$. Notice that this line in the $(i, j)$–plane has slope $m = 1 - \frac{2}{t}$. For $t'$ close to $t$ and $t' < t$, we have

$$\Upsilon_K(t') = -2i + (i - j)t'$$

for one of those pairs $(i, j)$. If $t'$ is near $t$ and $t' > t$, then

$$\Upsilon_K(t') = -2i' + (i' - j')t'$$

for another of these pairs, $(i', j')$, which may be the same. Notice that these are equal at $t$, giving the continuity of $\Upsilon_K(t)$.

We now see that a singularity of $\Upsilon_K(t)$ occurs if $j - i \neq j' - i'$. With these observations, the proofs of (1), (2) and (3) are complete.

For (4), our computations have shown that the change in $\Upsilon'_K(t)$, denoted $\Delta\Upsilon'_K(t)$, is given by $\Delta\Upsilon'_K(t) = (j - j') - (i - i')$ for some appropriate $(i, j)$ and $(i', j')$. Since both are assumed to lie on a line of slope $1 - \frac{2}{t}$, we have $j - j' = \left(1 - \frac{2}{t}\right)(i - i')$, so

$$\Delta\Upsilon'_K(t) = \left(1 - \frac{2}{t}\right)(i - i') - (i - i') = -\frac{2}{t}(i - i').$$

This completes the proof of the theorem. □

**Corollary 8.2** *For any knot $K$ and for $t = \frac{p}{q}$ with $\gcd(p, q) = 1$,*

$$\frac{t}{2}\Delta\Upsilon'_K(t) = kp,$$

*where $k$ is some integer if $p$ is odd, or half-integer if $p$ even.*

**Proof** By Theorem 8.1(4), $\left|\frac{t}{2}\Delta\Upsilon'_K(t)\right| = |i - i'|$ for some pair of integers $i$ and $i'$, where there are two lattice points on a line of slope $m = 1 - \frac{2}{t}$. Thus, we want to constrain the possible differences between the first coordinates of such lattice points.

For $t = \frac{p}{q}$, we have $m = -(2q - p)/p$. Since $\gcd(p, q) = 1$, in reduced terms this is either $m = -(2q - p)/p$ or $m = -\left(q - \frac{p}{2}\right)/\frac{p}{2}$ if $p$ is odd or even, respectively. Two lattice points on such a line have first coordinates differing by a multiple of $p$ or of $\frac{p}{2}$ if $p$ is odd or even, respectively. The completes the proof. □

# 9 The three-genus, $g_3(K)$

**Theorem 9.1** $|\Upsilon'_K(t)| \leq g_3(K)$ *for nonsingular points of* $\Upsilon'_K(t)$.

**Proof** According to [8], if $K$ is of genus $g$, then all elements of $\mathrm{CF}(K)$ have filtration level $(i, j)$, where

$$-g \leq i - j \leq g.$$

It follows immediately from Theorem 8.1(2) that $|\Upsilon'_K(t)| \leq g_3(K)$. □

We also observe that the genus of $K$ constrains the possible points of singularity of $\Upsilon'_K(t)$.

**Theorem 9.2** *Suppose that* $\Upsilon'_K(t)$ *has a singularity at* $t = \frac{p}{q}$*, with* $\gcd(p, q) = 1$. *Then:*

- *If* $p$ *is odd,* $q \leq g_3(K)$.
- *If* $p$ *is even,* $q \leq 2g_3(K)$.

**Proof** Suppose that a line of slope $m = -\frac{a}{b}$, where $0 < b < a$, contains two distinct points of the form $(i, j)$ with $|i - j| \leq g_3(K)$. It follows quickly that the genus bound implies

$$a \leq 2g_3(K) - b.$$

To express this in terms of $t$, suppose $t = \frac{p}{q}$ with $\gcd(p, q) = 1$. Then

$$m = 1 - \frac{2}{t} = -\frac{2q - p}{p}.$$

If $p$ is odd, then $\gcd(2q - p, p) = 1$. If $p$ is even, say $p = 2k$, then $\gcd(2q - p, p) = \gcd(2q, p) = 2$ and $m = -(q - k)/k$, with $q$ and $k$ relatively prime.

In the first case, with $p$ odd, we have $2q - p \leq 2g_3(K) - p$, so $q \leq g_3(K)$.

In the second case, with $p$ even, we have $q - k \leq 2g_3(K) - k$, so $q \leq 2g_3(K)$. □

# 10 $\Upsilon_K(t)$ as a knot concordance invariant

If knots $K_1$ and $K_2$ are concordant, then there is an equality among $d$–invariants: $d(S^3_N(K_1), \mathfrak{s}_m) = d(S^3_N(K_2), \mathfrak{s}_m)$ for all $N \in \mathbb{Z}$ and $m \in \mathbb{Z}$ with $-\frac{N-1}{2} \leq m \leq \frac{N-1}{2}$. Here $S^3_N(K)$ denotes $N$ surgery on $K$, $d$ is the Heegaard Floer correction term, and $\mathfrak{s}_m$ is a Spin$^c$ structure, with $m$ given by a specific enumeration of Spin$^c$ structures;

all are described in [6]. (In the case that $N$ is odd, this range of $m$ includes all possible $\mathrm{Spin}^c$ structures.)

If $N$ is large, then $d(S^3_N(K_1), \mathfrak{s}_0) = D(K) + S(N)$, where $D(K)$ is the largest grading of a class $z$ in the homology of $\mathrm{CF}(K)_{\{i\le 0, j\le 0\}}$ for which $U^k z$ is nontrivial for all $k > 0$, and $S(N)$ is some rational function defined on the integers, independent of $K$.

In the case that $K$ is slice, we see that the maximal grading $D(K) = D(u)$, where $u$ is the unknot. This implies that $D(K) = 0$ for a slice knot $K$. We have a nesting of complexes

$$\mathrm{CF}(K)_{\{i\le 0,\, j\le 0\}} \subset (\mathrm{CF}(K), \mathcal{F}_t)_0.$$

Since $(0,0)$ is at $\mathcal{F}_t$ filtration level $0$, it follows that $\nu(\mathrm{CF}(K), \mathcal{F}_t) \le 0$; thus $\Upsilon_K(t) \ge 0$.

However, $-K$ is also slice, so $-\Upsilon_K(t) \ge 0$. It follows that $\Upsilon_K(t) = 0$. An additive invariant of knots that vanishes on slice knots is a concordance invariant.

## 11 The concordance-genus

The concordance-genus $g_c(K)$ of a knot $K$, defined in [4], is the minimal genus among all knots concordant to $K$. Since $\Upsilon_K(t)$ is a concordance invariant, the genus bounds in Section 9 apply to the concordance genus.

**Theorem 11.1** *For all nonsingular points of $\Upsilon_K(t)$, $|\Upsilon'_K(t)| \le g_c(K)$. The jumps in $\Upsilon'_K(t)$ occur at rational numbers $\frac{p}{q}$. For $p$ odd, $q \le g_c(K)$. If $p$ is even, $\frac{q}{2} \le g_c(K)$.*

## 12 Bounds on the four-genus, $g_4(K)$

Let $\mathrm{CF}(K)_{0,m}$ denote the bifiltered subcomplex $\mathrm{CF}(K)_{\{i\le 0,\, j\le m\}}$. We let $\nu^-(K)$ denote the minimum value of $m$ such that the homology of $\mathrm{CF}(K)_{0,m}$ contains a nontrivial grading $0$ element of the homology of $\mathrm{CF}(K)$, which we recall is isomorphic to $\Lambda$ with $1$ at grading $0$. There is the following result of Hom and Wu [1], built from work of Rasmussen [10]. (In [1] the invariant $\nu^+$ is described; the equivalence with $\nu^-$ is presented in [5].)

**Proposition 12.1** [1, Proposition 2.4] $\nu^- \le g_4(K)$.

Based on this, we show that $\Upsilon_K(t)$ provides a bound on $g_4(K)$.

**Theorem 12.2** $|\Upsilon_K(t)| \leq t g_4(K)$ *for all* $t \in [0, 2]$.

**Proof** Since $(0, m)$ is at $\mathcal{F}_t$ filtration level $\frac{tm}{2}$, we have the containment

$$\mathrm{CF}(K)_{0,m} \subset (\mathrm{CF}(K), \mathcal{F}_t)_{tm/2}.$$

Since $\mathrm{CF}(K)_{0,\nu^-}$ contains an element of grading 0 in the homology of $\mathrm{CF}(K)$, so does the subcomplex $(\mathrm{CF}(K), \mathcal{F}_t)_{t\nu^-/2}$. Thus, $\nu(\mathrm{CF}(K), \mathcal{F}_t) \leq \frac{1}{2} t \nu^-$. By the previous proposition, $\nu(\mathrm{CF}(K), \mathcal{F}_t) \leq \frac{1}{2} t g_4(K)$.

Considering $-K$, we have $\nu(\mathrm{CF}(-K), \mathcal{F}_t) \leq \frac{1}{2} t g_4(-K)$; it follows that

$$-\nu(\mathrm{CF}(K), \mathcal{F}_t) \leq \tfrac{1}{2} t g_4(K).$$

Combining these yields

$$|\nu(\mathrm{CF}(K), \mathcal{F}_t)| \leq \tfrac{1}{2} t g_4(K).$$

Multiplying by $-2$ yields the desired conclusion. □

# 13 Crossing change bounds

Here we sketch a proof of [5, Proposition 1.10]. The argument is essentially the same as used in [3] to prove the corresponding fact about $\tau(K)$.

**Theorem 13.1** *Let* $K_-$ *and* $K_+$ *be knots with identical diagrams, except at one crossing which is either negative or positive, respectively. Then, for* $t \in [0, 1]$,

$$\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + t.$$

**Proof** First note that $K_- \# -K_+$ can be changed into the slice knot $K_+ \# -K_+$ by changing a negative crossing to positive. Thus, $g_4(K_- \# -K_+) \leq 1$. It follows that

$$-t \leq \Upsilon_{K_-}(t) - \Upsilon_{K_+}(t) \leq t. \tag{13-1}$$

Next, note that $K_- \# -K_+ \# \, T(2,3)$ can be changed into the slice knot $K_+ \# -K_+$ by changing one negative crossing to positive and one positive crossing to negative. Thus, it too has four-genus at most 1: it bounds a singular disk with two singularities of opposite sign, and these can be tubed together. A simple computation for $T(2,3)$ yields $\Upsilon_{T(2,3)}(t) = -t$ for $0 \leq t \leq 1$. Thus,

$$-t \leq \Upsilon_{K_-}(t) - \Upsilon_{K_+}(t) - t \leq t,$$

which we rewrite as

$$0 \leq \Upsilon_{K_-}(t) - \Upsilon_{K_+}(t) \leq 2t. \tag{13-2}$$

Combining (13-1) and (13-2),

$$0 \le \Upsilon_{K_-}(t) - \Upsilon_{K_+}(t) \le t.$$

Adding $\Upsilon_{K_+}(t)$ to all terms yields the desired conclusion,

$$\Upsilon_{K_+}(t) \le \Upsilon_{K_-}(t) \le \Upsilon_{K_+}(t) + t. \qquad \square$$

**Note** This argument can be easily modified to show that if there is a singular concordance from $K$ to $J$ with a single positive double point, then $\Upsilon_K(t) \le \Upsilon_J(t) \le \Upsilon_K(t) + t$.

## 14 The Ozsváth–Szabó $\tau$–invariant and $\Upsilon_K(t)$ for small $t$

For small $t$, $\Upsilon_K(t)$ is determined by the $\tau$ invariant defined in [7]. We review the definition below. Here is the statement of the result:

**Theorem 14.1** *For $t$ small, $\Upsilon_K(t) = -\tau(K)t$.*

The subquotient complex $\mathrm{CF}(K)_{\{i\le 0\}}/\mathrm{CF}(K)_{\{i<0\}}$ will be denoted $\widehat{\mathrm{CF}}(K)$. (Usually, $\widehat{\mathrm{CF}}$ is written $\widehat{\mathrm{CFK}}$.) It is filtered by the Alexander filtration and has homology $\mathbb{F}$, supported in grading $0$. The invariant $\tau(K)$ is defined to be the least integer $\tau$ such that the map on homology $H_0(\widehat{\mathrm{CF}}(K)_{\{j\le\tau\}}) \to H_0(\widehat{\mathrm{CF}}(K)) \cong \mathbb{F}$ is surjective.

We wish to relate $\tau(K) = \tau$ to an invariant of $\mathrm{CF}(K)$. The needed technical result is the following:

**Lemma 14.2** *If $\tau(K) = \tau$, then there is a cycle $w \in \mathrm{CF}(K)_{\{i\le 0,\, j\le\tau\}\cup\{i<0\}}$ representing a nontrivial element in $H_0(\mathrm{CF}(K))$.*

**Proof** From the definition of $\tau$ we see that there is a chain $x \in \mathrm{CF}(K)_{\{i\le 0,\, j\le\tau\}\cup\{i<0\}}$ that in the quotient $\widehat{\mathrm{CF}}(K)$ is a cycle that represents a generator of the homology group $H_0(\widehat{\mathrm{CF}}(K))$.

Since the chain $x$ represents a cycle in $\widehat{\mathrm{CF}}(K)$, it has the property that $\partial x = y$, where $y \in \mathrm{CF}(K)_{i<0}$. Note that $y$ is a cycle and $\mathrm{gr}(y) = -1$. Since $H_{-1}(\mathrm{CF}(K)_{i<0}) = 0$, there is a chain $z \in \mathrm{CF}(K)_{i<0}$ with $\partial z = y$. Thus, $x + z$ is a cycle in the complex $\mathrm{CF}(K)_{\{i\le 0,\, j\le\tau\}\cup\{i<0\}}$. The map $H_0(\mathrm{CF}(K)_{i\le 0}) \to H_0(\widehat{\mathrm{CF}}(K))$ is an isomorphism; both groups are isomorphic to $\mathbb{F}$. Thus, $x+z$ represents a generator of $H_0(\mathrm{CF}(K)_{i\le 0})$. The map $H_0(\mathrm{CF}(K)_{i\le 0}) \to H_0(\mathrm{CF}(K))$ is an isomorphism, completing the proof. $\square$

**Proof of Theorem 14.1** For $t$ small, we consider the filtration $\mathcal{F}_t$ and the filtration level $s = \frac{t}{2}\tau$. Then one has $\mathrm{CF}(K)_s = \mathrm{CF}(K)_{\{i \leq 0,\, j \leq \tau\} \cup \{i<0\}}$. By Lemma 14.2, this subcomplex contains a cycle that represents an element of grading $0$ in $H(\mathrm{CF}(K))$. Thus, for this $\mathcal{F}_t$ filtration, $\nu \leq \frac{t}{2}\tau$.

On the other hand, suppose that $\nu < \frac{t}{2}\tau$. Then there would exist a cycle

$$z \in \mathrm{CF}(K)_{\{i \leq 0,\, j \leq \tau - 1\} \cup \{i<0\}}$$

representing a generator of $H(\mathrm{CF}(K))$ of grading $0$. However, the image of $z$ in $\widehat{\mathrm{CF}}(K)$ would be an element in $\widehat{\mathrm{CF}}(K)_{\tau-1}$ that represents a generator of $H_0(\widehat{\mathrm{CF}}(K))$. But $\tau$ is by definition the lowest level at which this can occur. Thus, we see that $\nu = \frac{t}{2}\tau$.

To conclude, recall that $\Upsilon_K(t) = -2\nu$, so $\Upsilon_K(t) = -\tau(K)t$, as desired. □

**Note** With care, one can check that in this argument, the condition that $t$ be small can be made precise by requiring that $t < 1/g_3(K)$. Of course, once the result is established for some set of small $t$, then Theorem 9.2 provides the bound $t < 1/g_3(K)$.

# 15 Equivalence of definitions of $\Upsilon_K(t)$

In this section we explain why $\Upsilon_K(t)$ as defined here agrees with that of [5].

Beginning with $\mathrm{CF}(K)$, a new complex $t\mathrm{CF}(K)$ can be constructed as follows. As an $\mathbb{F}$–vector space,

$$t\mathrm{CF}(K) = \mathrm{CF}(K) \otimes_\Lambda \mathbb{F}[v^{1/n}, v^{-1/n}],$$

where $U$ acts on $\mathbb{F}[v^{1/n}]$ via multiplication by $v^2$. This has the structure of an $\mathbb{F}[v^{1/n}, v^{-1/n}]$–module. To simplify notation, we write $\Lambda' = \mathbb{F}[v^{1/n}, v^{-1/n}]$.

There are (rational) filtrations Alg and Alex on $t\mathrm{CF}(K)$ which are consistent with those on the $\Lambda$–submodule $\mathrm{CF}(K)$. The action of $v^{1/n}$ lowers filtration levels by $1/2n$. Thus, $U = v^2$ lowers filtration levels by $1$, as it should. Similarly, the Maslov grading $M(x)$ naturally extends to $t\mathrm{CF}(K)$ so that the action of $v^{1/n}$ lowers this grading by $1/n$, and thus $U = v^2$ continues to lower the Maslov grading by $2$.

There is a rational grading on $t\mathrm{CF}(K)$ defined via the Maslov grading, $M$, along with the algebraic and Alexander filtrations. If $x$ is an element at filtration level $(i, j)$, then:

(15-1) $$\mathrm{gr}_t(x) = M(x) - t(j-i).$$

(In [5], only generators at algebraic filtration level $0$ are used to define $\mathrm{gr}_t$, so $i = 0$ and the formula $\mathrm{gr}_t(x) = M(x) - t\,\mathrm{Alex}(x)$ is presented.) One checks that $U$ lowers

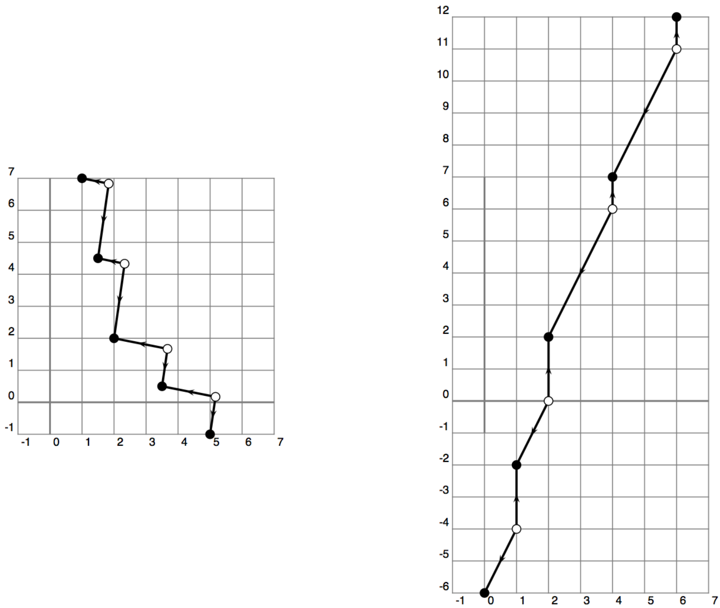


Figure 2: $t\mathrm{CF}_{t=1/3}(T(3,7))$ and $t\mathrm{CF}_{t=2}(T(3,7))$

$\mathrm{gr}_t$–gradings by $2$, so, on the extension to $t\mathrm{CF}(K)$, $v$ lowers gradings by $1$ and $v^{1/n}$ lowers gradings by $1/n$.

If $x$ is a filtered generator of $\mathrm{CF}(K)$ with $\partial x = \sum y_l$, then the boundary $\partial_t$ is defined so that $\partial_t x = \sum v^{\alpha_l} y_l \in t\mathrm{CF}(K)$, with the values of $\alpha_l$ given explicitly in [5]. This extends naturally to a boundary operator on all of $t\mathrm{CF}(K)$.

Given that the operator $\partial_t$ is well-defined, it is a simple matter to determine its value. Suppose that $x$ is a filtered generator of $\mathrm{CF}(K)$ at filtration level $(i,j)$, Maslov grading $g$, and suppose also that $\partial x = \sum y_l$. Let $y$ denote one of the terms in this sum, at filtration level $(i',j')$, necessarily of grading $g-1$. Then $x$, viewed as an element of $t\mathrm{CF}$, is of grading $g - t(j-i)$, and $y$ has grading $g-1-t(j'-i')$. In $\partial_t x$, the term $v^{\alpha} y$ appears, and $\alpha$ is such that $\mathrm{gr}_t(v^{\alpha} y) = \mathrm{gr}_t(x) - 1$. Rewriting this, we have $(g-1) - t(j'-i') - \alpha = g - t(j-i) - 1$. That is,

$$\alpha = t((j-j') - (i-i')). \tag{15-2}$$

As two examples, Figure 2 illustrates the complexes $t\mathrm{CF}(K)$ for $K = T(3,7)$, with $t = \frac{1}{3}$ and $t = 2$. The construction is straightforward using (15-1) and the fact that $v$ shifts along the diagonal a distance of $\frac{1}{2}$ down and to the left. The portion of the complex illustrated was chosen because its homology is $\mathbb{F}$ in grading $0$ and represents the generator of the homology of $t\mathrm{CF}$ in grading $0$. In the case that $t = \frac{1}{3}$, the full complex consists of the illustrated complex along with all its translates a distance $\frac{k}{6}$ for $k \in \mathbb{Z}$ along the diagonal. In the case of $t = 2$, the translates are those a distance $\frac{k}{2}$ along the diagonal.

It is apparent from these examples that the Alexander filtration is not a filtration of the chain complex, since some arrows increase the Alexander filtration level. However, as is easily verified, the algebraic filtration is a filtration on the chain complex.

**Definition 15.1** For $t = \frac{m}{n}$, denote by $t\mathrm{CFK}^-(K)$ the complex $t\mathrm{CF}(K)_{i \le 0}$.

**Note** In [5], this complex is denoted $t\mathrm{CFK}(K)$. In fact, it is the complex that is explicitly constructed. Here we first introduced the infinity complex to be consistent with our earlier constructions.

**Definition 15.2** For $t = \frac{m}{n}$, let $\Upsilon_K(t)$ be the maximal grading of a class in the homology of $t\mathrm{CFK}^-(K)$ that maps to a nontrivial element in the homology of $t\mathrm{CF}(K)$. Equivalently, it is the maximal grading of a class in the homology of $t\mathrm{CFK}^-(K)$ which is not in the kernel of $v^k$ for all $k > 0$.

**Lemma 15.3** *The value of $\Upsilon_K(t)$ as just defined is equal to $-2s$, where $s$ is the least number for which the homology of $t\mathrm{CF}(K)_{i \le s}$ contains an element of grading $0$ that represents a nontrivial element of the homology of $t\mathrm{CF}(K)$.*

**Proof** This follows from a simple change of coordinates. □

## 15.1 The two definitions of $\Upsilon_K(t)$ agree

Suppose that, using this definition of $\Upsilon_K(t)$, we have $\Upsilon_K(t) = -2s$. This implies that $t\mathrm{CF}(K)_{i \le s}$ contains a cycle $z$ representing a nontrivial generator of grading 0 in the homology of $t\mathrm{CF}(K)$. Write $z = \sum x_l$, where the $x_l$ are filtered generators. Some $x_l$ has filtration level $(s, j)$, and none of the $x_l$ has algebraic filtration level greater than $s$.

From the regrading formula given in (15-1), $\mathrm{gr}_t(x) = M(x) - t(j-i)$, we see that generators of $\mathrm{CF}(K)$ at filtration level $(i, j)$ and grading 0 yield generators of grading 0 in $t\mathrm{CF}(K)$ at filtration level $\left(i + \frac{t}{2}(j-i), j + \frac{t}{2}(j-i)\right)$. (Recall that shifting down and to the left by $t$ units decreases the grading by $2t$.) We are thus led to consider the transformation

$$(i, j) \mapsto \left(\left(1 - \frac{t}{2}\right)i + \frac{t}{2}j, -\frac{t}{2}i + \left(1 + \frac{t}{2}\right)j\right).$$

Its inverse is given by

$$(i, j) \mapsto \left(\left(1 + \frac{t}{2}\right)i - \frac{t}{2}j, \frac{t}{2}i + \left(1 - \frac{t}{2}\right)j\right).$$

Under this transformation, for a fixed value of $s$, the vertical line $\{(s,z) \mid z \in \mathbb{R}\}$ is carried to the line (in the CF$(K)$–plane) $\left\{\left(\left(1+\frac{t}{2}\right)s - \frac{t}{2}z, \frac{t}{2}s + \left(1-\frac{t}{2}\right)z\right) \mid z \in \mathbb{R}\right\}$. Relabeling the coordinate system $(x,y)$, this is the line

$$y = \left(1 - \frac{2}{t}\right)x + \frac{2}{t}s.$$

Comparing with (5-1), we see that the homology of the filtered complex $(\mathrm{CF}(K), \mathcal{F}_t)_s$ contains a generator of grading $0$ that is nontrivial in the homology of CF$(K)$, and that this is not the case for $(\mathrm{CF}(K), \mathcal{F}_t)_{s'}$ for any $s' < s$. Thus, the value of $\Upsilon_K(t)$ as defined in Section 5 is $-2s$, and the definitions agree.

# Appendix: A structure theorem for CF$(K)$

In [2, Chapter 11], vertical and horizontal reductions of CF$(K)$ are discussed. That presentation applies to the filtered complex $(\mathrm{CF}(K), \mathcal{F}_t)$, but adjustments in the details would be required because, for instance, the horizontal and vertical filtrations are integer-valued rather than being real filtrations. Since the argument in the present case is straightforward, we present it in detail.

Viewed as a $\Lambda$–module, CF$(K)$ is freely generated by a finite set $\{w_i\}_{1 \le i \le m}$. We again simplify notation by suppressing the indexing set and write $\{w_i\}$. This set can be chosen so that the set $\{U^k w_i\}_{k \in \mathbb{Z}}$ forms a bifiltered graded basis for the $\mathbb{F}$–complex CF$(K)$. We will refer to any such set $\{w_i\}$ as a $\Lambda$–basis for CF$(K)$. A $\Lambda$–module change of basis among the $w_i$ that preserves gradings and filtration levels induces a change of bifiltered graded basis for the $\mathbb{F}$–complex CF$(K)$. We will refer to any such change of basis as a $\Lambda$–change of basis of CF$(K)$. Analogous notation will be used when working with the filtered graded complex $(\mathrm{CF}(K), \mathcal{F}_t)$.

**Theorem A.1** *Let $t \in [0,2]$. As a $\Lambda$–module,* CF$(K)$ *has a basis $\{\alpha, \beta_1, \dots, \beta_k\}$, inducing a splitting of* CF$(K)$ *(as a $\Lambda$–module) as the direct sum* CF$(K) \cong \mathcal{T} \oplus \mathcal{A}$*, where $\mathcal{T}$ is freely generated by $\alpha$ and $\mathcal{A}$ is freely generated by $\{\beta_1, \dots, \beta_k\}$. This splitting has the following properties:*

- $(\mathrm{CF}(K), \mathcal{F}_t) \cong \mathcal{T} \oplus \mathcal{A}$ *as a filtered graded $\mathbb{F}$–complex.*
- *The complex $\mathcal{T}$ has filtered graded basis $\{U^k \alpha\}_{k \in \mathbb{Z}}$, the boundary map is trivial on $\mathcal{T}$, and* $\mathrm{gr}(\alpha) = 0$.
- *The complex $\mathcal{A}$ has filtered graded basis $\{U^k \alpha_i\}_{k \in \mathbb{Z}}$ and has trivial homology: $H(\mathcal{A}) = 0$.*

**Proof** We begin with the $\Lambda$–generating set of CF$(K)$, $\{w_i\}$.

By replacing generators with their $U^k$ translates and renaming the generators, we can decompose this into two subsets: $\{x_i\}$, all of grading $0$, and $\{y_i\}$, all of grading $1$.

To simplify notation, we abbreviate the filtered graded $\mathbb{F}$–complex $(\mathrm{CF}(K), \mathcal{F}_t)$ by $\mathrm{CF}_t$.

(1) Let $A$ be a cycle in $\mathrm{CF}_t$ having the least filtration level among cycles representing nontrivial classes in $H_0(\mathrm{CF}_t)$. After reordering the generators, we can write $A = x_1 + \cdots + x_k$, with the filtration levels nonincreasing. Replacing $x_1$ with $x_1 + \cdots + x_k$ as the first generating element (over $\Lambda$) induces a filtered change of basis for $\mathrm{CF}_t$. Thus, the first element of the $\Lambda$–basis, which we now denote $A_1$, is a cycle of least filtration level representing a nontrivial element of $H_0(\mathrm{CF}_t)$.

(2) Consider the set of all generating elements $y_i$ that have the property that $A_1$ is a component of $\partial y_i$. After reordering the basis, we can assume these are $\{y_1, y_2, \dots, y_k\}$ for some $k$, and that the filtrations are in nondecreasing order. Make the $\Lambda$–change of basis that replaces each $y_i$ for $2 \le i \le k$ with $y_i + y_1$. This induces a filtered change of basis of $\mathrm{CF}_t$. Now, the only generator having $A_1$ as a component of its boundary is $y_1$, which we relabel $B_1$.

(3) After perhaps reordering the $x_i$, we have either

$$\partial B_1 = A_1 \quad \text{or} \quad \partial B_1 = A_1 + x_2 + \cdots + x_k$$

for some $k \ge 2$, with the filtration levels nonincreasing. Since $\partial^2 = 0$, it follows that $B_1$ is not a component of any element in the image of $\partial$.

If $\partial B_1 = A_1$, then we see that $\{A_1, B_1\}$ generates an acyclic *summand* of $\mathrm{CF}_t$, and thus $A_1$ would not represent a nontrivial element in homology.

We have $\partial B_1 = A_1 + x_2 + \cdots x_k$ for some $k \ge 2$. Make the $\Lambda$–change of basis that replaces $x_2$ with $x_2 + \cdots x_k$, now calling this new element $A_2$. Then $\partial B_1 = A_1 + A_2$. Note that since $A_1$ is a cycle and $A_1 + A_2 = \partial B_1$ is a cycle, that $A_2$ is a cycle representing the same homology class as $A_1$. Hence the filtration level of $A_2$ is greater than or equal to that of $A_1$.

(4) We now repeat the previous argument, making a change of basis so that the only basis elements with boundary that include $A_2$ as a component are $B_1$ and perhaps a second generator, which we denote $B_2$.

(5) This step-by-step procedure must eventually stop, at which time there is constructed a summand of the $\mathbb{F}$–complex $\mathrm{CF}_t$,

$$D = A_1 \leftarrow B_1 \rightarrow A_2 \leftarrow B_2 \rightarrow A_3 \leftarrow \cdots \rightarrow B_{k-1} \rightarrow A_k.$$

Note that the process must end with an $A_k$; if it stopped with a $B_k$, the resulting complex would be acyclic and thus not contain a nontrivial element in homology. This complex is a summand of the complex $\mathrm{CF}_t$. Note that $\Lambda D$ is a summand of a direct sum decomposition of $\mathrm{CF}_t$, as a subcomplex and also as a submodule of the $\Lambda$–module.

(6) Since $A_1$ has the lowest filtration level among the $A_i$, we can replace each $A_i$ with $A_1 + A_i$ to form a new basis. The complex then splits in the following way:

$$A_1 \oplus \left[B_1 \to (A_1{+}A_2) \leftarrow B_2 \to (A_1{+}A_3) \leftarrow \cdots \to B_{k-1} \to (A_1{+}A_k)\right].$$

We let $\mathcal{T} = \Lambda A_1$. It satisfies the required conditions of the theorem. Since, as a $\Lambda$–module, $H(\mathcal{T}) \cong H(\mathrm{CF}_t)$, the complementary summand to $\mathcal{T}$ must be acyclic. That complementary summand yields the summand $\mathcal{A}$ in the statement of the theorem. □

## References


[1] **J Hom**, **Z Wu**, *Four-ball genus bounds and a refinement of the Ozváth–Szabó tau invariant*, J. Symplectic Geom. 14 (2016) 305–323 MR

[2] **R Lipshitz**, **P Ozsváth**, **D Thurston**, *Bordered Heegaard Floer homology: invariance and pairing*, preprint (2008) arXiv

[3] **C Livingston**, *Computations of the Ozsváth–Szabó knot concordance invariant*, Geom. Topol. 8 (2004) 735–742 MR

[4] **C Livingston**, *The concordance genus of knots*, Algebr. Geom. Topol. 4 (2004) 1–22 MR

[5] **P Ozsváth**, **A Stipsicz**, **Z Szabó**, *Concordance homomorphisms from knot Floer homology*, preprint (2014) arXiv

[6] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR

[7] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and the four-ball genus*, Geom. Topol. 7 (2003) 615–639 MR

[8] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004) 311–334 MR

[9] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR

[10] **J A Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University, Cambridge MA (2003) MR Available at `http://search.proquest.com/docview/305332635` A version is available on arXiv



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