

The Asaeda–Haagerup fusion categories

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This paper is dedicated to the memory of Uffe Haagerup.

Abstract. The classification of subfactors of small index revealed several new subfactors. The first subfactor above index 4, the Haagerup subfactor, is increasingly well understood and appears to lie in a (discrete) infinite family of subfactors where the $\mathbb{Z}/3\mathbb{Z}$ symmetry is replaced by other finite abelian groups. The goal of this paper is to give a similarly good description of the Asaeda–Haagerup subfactor which emerged from our study of its Brauer–Picard groupoid. More specifically, we construct a new subfactor \mathcal{S} which is a $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ analogue of the Haagerup subfactor and we show that the even parts of the Asaeda–Haagerup subfactor are higher Morita equivalent to an orbifold quotient of \mathcal{S} . This gives a new construction of the Asaeda–Haagerup subfactor which is much more symmetric and easier to work with than the original construction. As a consequence, we can settle many open questions about the Asaeda–Haagerup subfactor: calculating its Drinfeld center, classifying all extensions of the Asaeda–Haagerup fusion categories, finding the full higher Morita equivalence class of the Asaeda–Haagerup fusion categories, and finding intermediate subfactor lattices for subfactors coming from the Asaeda–Haagerup categories. The details of the applications will be given in subsequent papers.

1. Introduction

A major motivation for classifying mathematical objects is to produce interesting new examples which would not have been found without an extensive search. For example, it was the classification of simple Lie algebras which uncovered E_8 and the classification of finite simple groups which revealed the Monster group. Similarly, a major motivation in the classification of small index subfactors is to find new interesting examples.

To any finite index finite depth subfactor $N \subset M$ one can assign a rich algebraic invariant called its standard invariant. One way to describe the standard invariant is that it consists of

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a pair of a unitary fusion category \mathcal{C} (consisting of certain N – N bimodules and called the principal even part) and an algebra object $A \in \mathcal{C}$ (which is the bimodule M). Furthermore, for finite index finite depth subfactors of the hyperfinite II_1 factor, the standard invariant is a complete invariant [22]. Almost all known unitary fusion categories can be constructed from finite groups or from quantum groups at roots of unity. Indeed, all subfactors of index less than 4 have standard invariants coming from quantum $\text{SU}(2)$ via conformal inclusions, and all finite depth subfactors of index equal to 4 have standard invariants which come from finite subgroups of $\text{SU}(2)$. It is natural to wonder whether the subfactors of index between 4 and 5 come from similar constructions.

There are five pairs of finite index finite depth subfactor standard invariants of index between 4 and 5:

- the Haagerup subfactor (and its dual),
- the Asaeda–Haagerup subfactor (and its dual),
- the extended Haagerup subfactor (and its dual),
- the Goodman–de la Harpe–Jones 3311 subfactor (and its dual),
- and the self-dual Izumi 2221 subfactor (and its complex conjugate).

The last two of these pairs come from quantum groups in an appropriate sense. The remaining three seem to not be related to finite groups or quantum groups via well-understood constructions. Thus they are very important examples for further study since they could be “exotic” or “exceptional.”

Of these three examples, the Haagerup subfactor is by far the best understood. Three different constructions of this subfactor standard invariant have been given, each of a different flavor [2, 13, 21]. One of these constructions allows for practical calculations, for example of the S and T matrices of the Drinfeld center of the even part [13]. Furthermore, the principal even part of the Haagerup subfactor has a collection of invertible objects forming the group $\mathbb{Z}/3\mathbb{Z}$, and all non-invertible objects in this even part are in a single orbit of the action of $\mathbb{Z}/3\mathbb{Z}$. On the one hand, this suggests that there may be a sense in which the Haagerup subfactor can be constructed from the group $\mathbb{Z}/3\mathbb{Z}$ (or a closely related group like S_3 , see [5]), and on the other hand it suggests that the Haagerup subfactor could be generalized by replacing $\mathbb{Z}/3\mathbb{Z}$ by other finite abelian groups. We call such a subfactor a 3^A subfactor, since its principal graph is a star graph with $\#A$ arms where each arm has three edges. The second-named author determined a numerical invariant for such subfactors given by solutions to certain polynomial equations, and by solving these equations he constructed a $3^{\mathbb{Z}/5\mathbb{Z}}$ subfactor [13]. Evans–Gannon constructed 3^G subfactors for several other cyclic groups of odd order by solving the polynomial equations using symmetries of the associated modular data [5]. Although there are still only finitely many examples known, it appears that the Haagerup subfactor may not be exceptional after all.

The main goal of this paper is to establish a similar story for the previously mysterious Asaeda–Haagerup subfactor. The basic philosophy of our approach is that one should never study a single subfactor at a time, but instead should study all the subfactors which come from inclusions of algebra objects $A \subset B$ in a fixed unitary fusion category \mathcal{C} . In other words, one fixes a finite collection of bimodules over a fixed factor R and looks at all subfactors $N \subset M$, where each of N and M can be built as direct sums of these particular bimodules. All of these subfactors fit together into a richer structure known as the maximal atlas or the Brauer–Picard

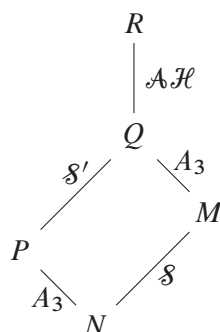


Figure 1. Our construction of the Asaeda–Haagerup subfactor can be summarized by the above inclusions of factors, where \mathcal{S} denotes our new index $5 + \sqrt{17}$ subfactor, \mathcal{S}' is its orbifold quotient, $\mathcal{A}\mathcal{H}$ is the Asaeda–Haagerup subfactor of index $\frac{5+\sqrt{17}}{2}$, and A_3 denotes index 2 inclusions.

groupoid [4]. This approach has two advantages. First, the combinatorics of this whole collection is richer and more restrictive than the combinatorics of the individual subfactors. For example, just by looking at fusion rules one can show that certain subfactors must exist which would be very difficult to construct directly, as illustrated in [10]. Second, it may be that some of the other subfactors in the maximal atlas are simpler than others. In this paper we emphasize the second point of view, and we relate the Asaeda–Haagerup subfactor to a simpler subfactor also appearing in its maximal atlas. That is, the complicated small index Asaeda–Haagerup subfactor is best thought of as a consequence of a simpler subfactor with larger index. This gives a new construction of the Asaeda–Haagerup subfactor, which still requires a difficult computation, but which is much more illuminating in terms of understanding this subfactor.

Here is a quick overview of our new construction of the Asaeda–Haagerup subfactor. We first construct a specific $3\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ subfactor \mathcal{S} by finding a solution to the appropriate polynomial equations, which were generalized to finite abelian groups of possibly even order in [14]. The subfactor \mathcal{S} has index $5 + \sqrt{17}$. Then we take an orbifold (or de-equivariantization) quotient of \mathcal{S} by adding an extra isomorphism between the objects corresponding to the trivial and non-trivial elements of $\mathbb{Z}/2\mathbb{Z}$. This new subfactor \mathcal{S}' has index $5 + \sqrt{17}$, and we call its even part $\mathcal{A}\mathcal{H}_4$. We then explicitly calculate the fusion rules for the dual even part of this subfactor (i.e. the bimodules over the larger factor) and see that it contains an object X with dimension $\frac{3+\sqrt{17}}{2}$ such that $X \otimes X \cong 1 \oplus X \oplus Y$ for some non-invertible simple object Y . By a simple skein theory argument [10, Theorem 3.4], the object $1 \oplus X$ has a unique algebra structure and this algebra gives a new subfactor with index $\frac{5+\sqrt{17}}{2}$. This subfactor must be the Asaeda–Haagerup subfactor $\mathcal{A}\mathcal{H}$, since it is easy to see that there is at most one finite depth subfactor of index $\frac{5+\sqrt{17}}{2}$. Altogether, the Asaeda–Haagerup subfactor comes as a natural consequence of the more symmetric and easier to understand subfactor \mathcal{S} .

In particular, we see that $\mathcal{A}\mathcal{H}_4$ lies in the same higher Morita equivalence class as the even parts of the Asaeda–Haagerup subfactor $\mathcal{A}\mathcal{H}_1$ and $\mathcal{A}\mathcal{H}_2$. Many invariants of fusion categories, most notably the Drinfeld center, are invariant under higher Morita equivalence. Thus instead of doing a calculation in $\mathcal{A}\mathcal{H}_1$ or $\mathcal{A}\mathcal{H}_2$, one can instead do the calculation in $\mathcal{A}\mathcal{H}_4$. In particular, in a follow-up paper we will give an explicit description of the Drinfeld center of the Asaeda–Haagerup fusion categories (a question which had remained open for fifteen years) by following the method in [13]. Similarly, since Etingof–Nikshych–Ostrik’s

classification of G extensions of fusion categories is Morita invariant, constructing extensions of \mathcal{AH}_4 yields extensions of the other fusion categories in its Morita class. More specifically, in another follow-up paper we can determine the homotopy type of the Brauer–Picard 3-groupoid of the Asaeda–Haagerup category, which splits as a product of Eilenberg–MacLane spaces $K(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, 1) \times K(\mathbb{C}^\times, 3)$. As a consequence, we get a plethora of interesting $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extensions of Asaeda–Haagerup fusion categories generalizing the constructions in [8], and we can use this to give a complete classification of all extensions of these fusion categories by any group.

It is natural to wonder why the Asaeda–Haagerup subfactor, which has only a 2-fold symmetry, should be related to an 8-fold symmetric $3^{\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ subfactor. In fact, we discovered this connection by working in the other direction: starting with the Asaeda–Haagerup subfactor and trying to describe its entire higher Morita equivalence class. An extensive combinatorial calculation suggested that \mathcal{AH}_4 was one of the only possible new tensor categories compatible with all the rich structure of the Asaeda–Haagerup subfactor. If \mathcal{AH}_4 does exist, one can then determine that it must have an equivariantization which would give a $3^{\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ subfactor. Please note that logically our construction does not require the extensive combinatorial calculations from [10], but the motivation to consider \mathcal{S} in the first place did come out of those calculations.

In fact, constructing \mathcal{AH}_4 has allowed us to complete our description of the higher Morita equivalence class of the Asaeda–Haagerup subfactors. Namely, there are exactly six fusion categories, and between any two of them there are exactly four Morita equivalences between them. As shown in [10], the group of Morita autoequivalences is the Klein 4-group. In fact, for \mathcal{AH}_4 all these autoequivalences are realized by outer automorphisms. As a consequence of this complete description of the higher Morita equivalence class, we are able to answer many other questions about the subfactors related to the Asaeda–Haagerup subfactors. For example, in a followup paper we will use this classification to find all lattices of intermediate subfactors related to Asaeda–Haagerup.

The outline of this paper is as follows. We begin in Section 2 with some background on fusion categories, algebra objects, the Brauer–Picard groupoid, and the Asaeda–Haagerup subfactor. This section includes some expository material summarizing the key ideas of our project of understanding subfactors via the Brauer–Picard groupoid.

In Section 3 we further describe the Brauer–Picard groupoid of the Asaeda–Haagerup fusion categories following [10]. In [10] we saw that there were at least three fusion categories in this Morita equivalence class, that there were exactly four bimodules between any two such fusion categories, and that the BP group was the Klein 4-group. Here we narrow things down further, showing that if \mathcal{AH}_4 exists, then there are exactly six fusion categories in the equivalence class. These calculations are combinatorial in nature and are quite similar to those in [10]. However, we would like to remind the reader that our new construction of \mathcal{AH} is *motivated* by the results of Section 3, but strictly speaking does not *depend* on the results of Section 3.

Section 4 is the main heart of the paper. We give a direct construction of \mathcal{S} as endomorphisms of an algebra using results of [14]. We then show that it has a de-equivariantization \mathcal{AH}_4 , and show that the Asaeda–Haagerup subfactor comes from an inclusion of algebras in \mathcal{AH}_4 .

In Section 5, we quickly sketch several applications of our main results. The full details of these applications will appear in later papers.

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2. Preliminaries

We begin with some background on fusion categories, subfactors, the Brauer–Picard groupoid, combinatorics of fusion rules, and the Asaeda–Haagerup subfactor. We assume that the reader is either familiar with the theory of fusion categories or the theory of subfactors. Thus the first subsection is aimed at readers who are familiar with subfactors but not with fusion categories, and the second subsection is aimed at readers familiar with fusion categories but not subfactors.

2.1. Fusion categories, module categories, and bimodule categories.

Definition 2.1 ([3]). A fusion category over an algebraically closed field k is a semisimple k -linear rigid monoidal category with finitely many simple objects and finite-dimensional morphism spaces such that the identity object is simple.

In this paper k will always be the field of complex numbers. An equivalence of fusion categories is a linear monoidal equivalence.

Example 2.2. Fix a finite collection of bifinite bimodules M_i over a II_1 factor A which are closed under tensor product in the sense that $M_i \otimes_A M_j \cong \bigoplus_k M_k^{\oplus n_{kj}}$. Then there is a fusion category whose objects are the direct sums of the M_i , the morphisms are bimodule maps, and the tensor product is \otimes_A .

A fusion category can be thought of as a higher analogue of an algebra, where instead of multiplying elements you tensor objects. Just as modules and bimodules play a crucial role in understanding algebras, module categories and bimodule categories play a similar role in understanding fusion categories.

Definition 2.3. A (left) module category \mathcal{M} over a fusion category \mathcal{C} is a category \mathcal{M} along with a biexact bifunctor from $\mathcal{C} \times \mathcal{M}$ to \mathcal{M} , along with a collection of unit and associativity isomorphisms satisfying certain coherence relations; see [20] for details. In this paper we will further assume that module categories over fusion categories are semisimple.

Example 2.4. If \mathcal{C} comes from a collection of bimodules over A as in the previous example, then one can build a left module category in the same way by finding a finite collection of simple A – B bimodules for a factor B which are closed under left multiplication by the A – A bimodules.

Similarly, one may define right module categories over fusion categories and bimodule categories over pairs of fusion categories. There is a natural notion of equivalence for module or bimodule categories.

Definition 2.5. A bimodule category over a pair of fusion categories $\mathcal{C}\mathcal{M}_{\mathcal{D}}$ is invertible if $\mathcal{C}\mathcal{M}_{\mathcal{D}} \boxtimes_{\mathcal{D}} \mathcal{D}\mathcal{M}^{\text{op}}\mathcal{C} \cong \mathcal{C}\mathcal{C}\mathcal{C}$ and $\mathcal{D}\mathcal{M}^{\text{op}}\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{C}\mathcal{M}_{\mathcal{D}} \cong \mathcal{D}\mathcal{D}\mathcal{D}$, where \mathcal{M}^{op} is the opposite bimodule category and $\boxtimes_{\mathcal{C}}$ and $\boxtimes_{\mathcal{D}}$ are the relative tensor products of bimodule categories; see [4] for details.

Two fusion categories are Morita equivalent if there is an invertible bimodule category between them; an invertible bimodule category is called a Morita equivalence.

Example 2.6. If $A \subset B$ is a finite depth subfactor, then the principal even part (certain A – A bimodules) and the dual even part (certain B – B bimodules) are Morita equivalent with the Morita equivalence given by a collection of A – B bimodules.

Example 2.7. If G is a finite group, let $\text{Vec}(G)$ be the category of G -graded vector spaces and $\text{Rep}(G)$ be the category of finite dimensional representations of G . Then Vec gives a Morita equivalence between $\text{Vec}(G)$ and $\text{Rep}(G)$, where the individual actions are given by forgetting and tensoring, but the associator for simple objects is

$$V = (1_g \otimes 1) \otimes V \rightarrow 1_g \otimes (1 \otimes V) = V,$$

with the map given by the action of g on V .

This Morita equivalence can be understood from the subfactor point of view by looking at the crossed product subfactor $N \subset M = N \rtimes G$ and thinking of $\text{Vec}(G)$ as N – N bimodules, Vec as N – M bimodules, and $\text{Rep}(G)$ as M – M bimodules.

Any simple module category over a fusion category $\mathcal{C}\mathcal{M}$ can be given the structure of an invertible bimodule category $\mathcal{C}\mathcal{M}_{\mathcal{D}}$, where $\mathcal{D} = (\mathcal{C}\mathcal{M})^*$ is the dual category of module endofunctors of $\mathcal{C}\mathcal{M}$. Conversely, for any invertible bimodule category $\mathcal{C}\mathcal{M}_{\mathcal{D}}$ we have $\mathcal{D} \cong (\mathcal{C}\mathcal{M})^*$.

Module categories can also be characterized in terms of algebras. This can be thought of as an algebraic substitute for subfactor theory where the role of factors is played by algebra objects.

Definition 2.8. An algebra in a fusion category is an object A together with a unit map $I \rightarrow A$ and a multiplication map $A \otimes A \rightarrow A$ satisfying the usual relations; see [20].

Example 2.9. If $N \subset M$ is a finite index finite depth subfactor, and \mathcal{C} is the category of N – N bimodules generated by M , then M itself is an algebra object in \mathcal{C} . In this case, saying that M is an algebra object means both that it is an algebra and that the multiplication map $M \otimes M \rightarrow M$ is a map of N – N bimodules.

In a similar way one can define modules over algebras in fusion categories. If A is an algebra in \mathcal{C} , then the category $A\text{-mod}$ of left A -modules in \mathcal{C} is a right module category over \mathcal{C} (although not necessarily semisimple), and similarly the category of right A -modules is a left module category.

An algebra A is called simple if its module category $A\text{-mod}$ is semisimple and indecomposable, and is called a division algebra if in addition A is simple as an A -module [10]. If A is a simple algebra in a fusion category, then the category $A\text{-mod-}A$ of A – A bimodules is the dual category of $A\text{-mod}$. In fact, every indecomposable module category arises this way. One can define the internal hom bifunctor $\underline{\text{Hom}}(M_1, M_2)$ from $\mathcal{C}\mathcal{M} \times \mathcal{C}\mathcal{M}$ to \mathcal{C} . The internal hom

satisfies $\text{Hom}(\underline{\text{Hom}}(M_1, M_2), X) \cong \text{Hom}(X \otimes M_1, M_2)$ for all $M_1, M_2 \in {}_{\mathcal{C}}\mathcal{M}$ and $X \in \mathcal{C}$, and the internal end $\underline{\text{End}}(M) = \underline{\text{Hom}}(M, M)$ is an algebra for every $M \in {}_{\mathcal{C}}\mathcal{M}$.

Theorem 2.10 ([20]). *Let $M \in {}_{\mathcal{C}}\mathcal{M}$ be a simple object in a simple module category over a fusion category. Then ${}_{\mathcal{C}}\mathcal{M}$ is equivalent to the category of modules over $\underline{\text{End}}(M)$ in \mathcal{C} .*

Example 2.11. If M comes from an A – B bimodule for two factors, then we have $\underline{\text{End}}(M) = M \otimes_B \bar{M}$, where \bar{M} is the contragredient B – A bimodule.

When M is a simple object, the algebra $\underline{\text{End}}(M)$ is a division algebra [10].

Theorem 2.12 ([10]). *The algebras $\underline{\text{End}}(X)$ and $\underline{\text{End}}(Y)$ for two objects X and Y in ${}_{\mathcal{C}}\mathcal{M}$ are isomorphic if and only if $X \cong Yg$ for some invertible object g in the dual category $({}_{\mathcal{C}}\mathcal{M})^*$.*

2.2. Subfactors. The theory of standard invariants of finite-index subfactors has been developed both for Type II_1 and for properly infinite factors. The classes of standard invariants which arise in the two settings are the same, so the choice of setting is in some sense a matter of taste; however, certain types of calculations or constructions may be easier in one setting or the other. In this paper we use infinite factors in order to utilize a construction of subfactors from endomorphisms of Cuntz algebras introduced by the second-named author in [11], which uses Type III factors.

There are two standard ways of building tensor categories related to an algebra A . The first is familiar algebraically, namely one considers a collection of bimodules over A which is closed under tensor product \otimes_A . When A is a von Neumann algebra, one needs to be a bit careful about whether one is considering Hilbert bimodules with the Connes fusion product or algebraic bimodules with the algebraic tensor product. Happily, in the finite index setting for II_1 factors, either choice leads to the same tensor category [15].

The second construction is called the category of sectors and is less familiar to algebraists. Nonetheless, it has a nice categorical description which we give here. If \mathcal{C} is a linear category, then there is a monoidal category whose objects are the automorphisms of \mathcal{C} and whose morphisms are natural transformations. The simplest kind of category is one with only one object M . In this setting the monoidal category of functors and natural transformations is called the category of sectors and written $\mathcal{C}(M)$.

If the underlying object is an algebra M , then $\mathcal{C}(M)$ can be described explicitly as follows. The objects are the endomorphisms of M and the morphisms are given by

$$\text{Hom}(\rho, \sigma) = \{v \in M : v\rho(x) = \sigma(x)v \text{ for all } x \in M\}$$

for $\rho, \sigma \in \text{End}(M)$. The composition of morphisms is given by multiplication in M , and the tensor product of objects is given by composition of endomorphisms. Finally, the tensor product of morphisms is given by the asymmetrical formula determined by composition of natural transformations, if $v \in (\rho, \sigma)$ and $u \in (\alpha, \beta)$, then $u \otimes v \in (\alpha \circ \rho, \beta \circ \sigma)$ is given by $\beta(v)u$.

If M is a Type III factor with separable predual, then any nice bimodule comes from an endomorphism, and so the category of sectors for M is especially appropriate. We use some standard definitions. We consider the category $\mathcal{C}_0(M)$ of finite-index unital normal $*$ -endomorphisms of M . A sector in $\mathcal{C}_0(M)$ is an isomorphism class of objects; the sector associated to

an endomorphism $\rho \in \text{End}_0(M)$ is denoted by $[\rho]$. We often simply write (ρ, σ) for $\text{Hom}(\rho, \sigma)$. For a self-dual endomorphism $\rho \in \text{End}_0(M)$, let \mathcal{C}_ρ be the tensor category tensor generated by ρ ; i.e. it is the full subcategory of $\mathcal{C}_0(M)$ whose objects are exactly those which are contained in some tensor power of ρ . We say that ρ has finite depth if \mathcal{C}_ρ has finitely many simple objects up to isomorphism; in this case \mathcal{C}_ρ is a fusion category.

We will often use sector notation for arbitrary tensor categories and bimodule categories, so objects are denoted by lowercase Greek letters, tensor product symbols are suppressed and $(\kappa, \lambda) := \dim(\text{Hom}(\kappa, \lambda))$.

Definition 2.13 ([16]). A Q-system is an algebra γ in $\mathcal{C}_0(M)$ such that the unit map $R \in (\text{id}, \gamma)$ is an isometry and the multiplication map $T \in (\gamma^2, \gamma)$ is a co-isometry.

If γ is a Q-system in $\mathcal{C}_0(M)$, then there is another properly infinite factor N and a pair of finite-index unital $*$ -homomorphisms $\iota : N \rightarrow M$ and $\bar{\iota} : M \rightarrow N$ such that $\bar{\iota}\iota = \gamma$. Then $\sigma = \bar{\iota}\iota$ is a Q-system in $\mathcal{C}_0(N)$. The homomorphisms $\rho^n \iota, n \in \mathbb{N}$ generate a subcategory \mathcal{M}_ι of the category $\text{Hom}_0(N, M)$ of finite-index homomorphisms from N to M (where morphisms are defined exactly as for End_0). Then \mathcal{M}_ι is an invertible \mathcal{C}_γ - \mathcal{C}_σ bimodule category, where the tensor product is given by composition. If γ is a finite-depth endomorphism, then \mathcal{C}_γ is a fusion category, \mathcal{M}_ι is equivalent to the category of $\gamma = \text{End}(\iota)$ modules in \mathcal{C}_γ , and \mathcal{C}_σ is then equivalent to the category of $\gamma - \gamma$ bimodules in \mathcal{C}_γ .

Conversely, if $N \subseteq M$ is a finite-index subfactor with inclusion map $\iota : N \rightarrow M$, then there is a dual homomorphism $\bar{\iota} : M \rightarrow N$ such that $\sigma = \bar{\iota}\iota$ is a Q-system in $\mathcal{C}_0(N)$, called the canonical endomorphism, and $\gamma = \bar{\iota}\iota$ is a Q-system in $\mathcal{C}_0(M)$, called the dual canonical endomorphism. The subcategories of $\mathcal{C}_0(N)$ and $\mathcal{C}_0(M)$ tensor generated by σ and γ are called the principal and dual even parts of the subfactor, respectively.

Finally, we recall the following calculation from [6]. Suppose $\gamma \cong \text{id} \oplus \rho$ is a finite-index endomorphism with ρ a self-conjugate irreducible endomorphism not isomorphic to the identity, and let $R \in (\text{id}, \rho^2)$ be an isometry. Then Q-systems for γ correspond to isometries $S \in (\rho, \rho^2)$ satisfying

$$\rho(S)R = SR, \quad \sqrt{d}R + (d-1)S^2 = \sqrt{d}\rho(R) + (d-1)\rho(S)S,$$

where $d = d(\rho) = [M : \rho(M)]^{\frac{1}{2}}$ is the statistical dimension of ρ . Two such Q-systems corresponding to S and S' are equivalent if and only if $S = \pm S'$.

2.3. The Brauer–Picard groupoid and related higher structures. There are several closely related higher categorical constructions important to the study of subfactors, which we describe in this section. All of them are motivated by trying to study not just a single division algebra in a fusion category \mathcal{C} , but all division algebras at once. Ocneanu called this the “maximal atlas” and it can be formalized in several ways.

The simplest construction is to look at all division algebras in \mathcal{C} (or, more generally, the Frobenius algebra objects in \mathcal{C}) and all bimodule objects between them. These bimodules form a 2-category as follows: the objects are division algebras in \mathcal{C} , the 1-morphisms are bimodules between division algebras, and the 2-morphisms are bimodule maps. The composition of 1-morphisms in this 2-category is given by the relative tensor product over the common algebra. Furthermore, this 2-category is rigid in the sense that every bimodule has a dual bimodule [23]. This rigidity is often called Frobenius reciprocity. This construction has a strong subfactor

flavor. Indeed, one can think of the objects of this 2-category as a certain finite collection of von Neumann algebras, the 1-morphisms as a finite collection of finite index bimodules between them which are closed under tensor product, and the 2-morphisms as bimodule maps.

The second construction looks beyond the original fusion category \mathcal{C} and thus is a little more difficult to understand from a pure subfactor point of view. Any division algebra in \mathcal{C} gives a Morita equivalence mod- A between the fusion category \mathcal{C} and the fusion category of A – A bimodules. Thus, it is natural to study all bimodule categories between all fusion categories in the Morita equivalence class of \mathcal{C} . On the other hand, different algebras can give rise to the same module category, so we can consider a structure that is not defined in terms of specific algebras.

Definition 2.14 ([4]). The Brauer–Picard 3-groupoid of a fusion category \mathcal{C} is the 3-groupoid whose objects are fusion categories Morita equivalent to \mathcal{C} , whose 1-morphisms are invertible bimodule categories between such fusion categories, whose 2-morphisms are equivalences between such bimodule categories, and whose 3-morphisms are isomorphisms of such equivalences.

This 3-groupoid can be truncated to an ordinary groupoid whose points are the fusion categories which are Morita equivalence to \mathcal{C} and whose arrows are equivalence classes of Morita equivalences. The group of Morita autoequivalences of a fusion category, considered modulo equivalence of bimodule categories, forms a group, called the Brauer–Picard group. The Brauer–Picard group is an invariant of the Morita equivalence class.

Typically, groupoids are considered up to equivalence. For example the trivial 1-point groupoid is equivalent to the groupoid with n points and exactly one arrow between any pair of them even though they have a different number of points. We often consider a finer invariant of the Brauer–Picard groupoid. Namely, consider the groupoid whose points are fusion categories up to ordinary equivalence (not Morita equivalence!) and whose arrows are equivalence classes of Morita equivalences. By Ocneanu rigidity [3], this gives a groupoid with a finite number of points. When we say something like the Brauer–Picard groupoid has exactly six points and exactly four arrows between each of them, we are referring to this refined version.

An autoequivalence of a fusion category \mathcal{C} is called inner if it is equivalent as a monoidal functor to conjugation by an invertible object in \mathcal{C} . The group $\text{Out}(\mathcal{C})$ of autoequivalences of a fusion category \mathcal{C} modulo inner autoequivalences is a subgroup of the Brauer–Picard group, via the map sending α to the bimodule ${}_{\mathcal{C}}\mathcal{C}_{\alpha(\mathcal{C})}$ where the right action is twisted by α .

Example 2.15. The Brauer–Picard groupoid of $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ has exactly one point. The group of outer automorphisms is the group of ordinary outer automorphisms of $\mathbb{Z}/p\mathbb{Z}$ acting in the obvious way. The Brauer–Picard group is the dihedral group $(\mathbb{Z}/p\mathbb{Z})^{\times} \rtimes \mathbb{Z}/2\mathbb{Z}$ where the subgroup is the group of outer automorphisms and the other coset comes from realizing Vec as a bimodule category over $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$. Such bimodules can be realized as follows: start with the Morita equivalence between $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ and $\text{Rep}(\mathbb{Z}/p\mathbb{Z})$ from Example 2.7 and then pick an equivalence $\text{Vec}(\mathbb{Z}/p\mathbb{Z}) \cong \text{Rep}(\mathbb{Z}/p\mathbb{Z})$.

Example 2.16. If G is a non-abelian group, then the Brauer–Picard groupoid of $\text{Vec}(G)$ has at least two distinct points since $\text{Vec}(G)$ and $\text{Rep}(G)$ are Morita equivalent via the bimodule from Example 2.7. Specifically, the Brauer–Picard groupoid of $\text{Vec}(S_3)$ has exactly two points,

and exactly two Morita equivalences between each points. The non-trivial Morita autoequivalence of $\text{Rep}(S_3)$ is given by $\text{Rep}(S_2)$ (this corresponds to the A_5 subfactor with index 3).

In general, there may be more than one equivalence ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}} \boxtimes_{\mathcal{D}} {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}} \cong {}_{\mathcal{C}}\mathcal{L}_{\mathcal{E}}$. In fact, such choices of equivalence are a torsor for π_2 of the Brauer–Picard groupoid. In particular, for the Asaeda–Haagerup subfactor there is no such ambiguity, since π_2 is trivial [8]. Similarly, if π_2 were non-trivial, then one has to be more careful about the associativity of these multiplication maps.

There is a third construction which unifies the above two points of view and elucidates the associativity of composition of bimodules. Although we will not use this third construction in this paper, it may clarify the above constructions conceptually. One can consider a 3-category whose objects are pairs $A \in \mathcal{C}$ of a Frobenius algebra object in a spherical fusion category, whose 1-morphisms are pairs ${}_A M_B \in {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ of a bimodule object in a bimodule category, whose 2-morphisms are pairs of a bimodule functor

$$\mathcal{F} : \mathcal{M} \rightarrow \mathcal{N}$$

and a bimodule map

$$f : \mathcal{F}(m) \rightarrow n,$$

and whose 3-morphisms are pairs of bimodule natural transformations satisfying a compatibility condition.

2.4. Combinatorics of fusion and module categories. A fusion category contains a lot of information. At the first level there is the combinatorial information of how the objects tensor, but then at the second level there is the linear algebraic information of all the morphisms and their compositions and tensor product. The combinatorial data is typically a rather weak invariant of the fusion category. However, once one considers the whole Brauer–Picard groupoid, the combinatorics for tensoring all the objects in all the tensor categories and all the bimodules is a lot richer.

The fusion ring of a fusion category \mathcal{C} is the based ring with basis indexed by the equivalence classes of simple objects of \mathcal{C} , with addition given by direct sum and multiplication given by tensor product. There is also an involution, defined on basis elements by duality in \mathcal{C} . There is a unique homomorphism from the fusion ring to \mathbb{R} which takes basis elements to positive numbers, called the Frobenius–Perron dimension, and denoted by d .

In a similar way, module categories and bimodule categories determine based modules and bimodules over the fusion rings of the corresponding fusion categories, which we call fusion modules and fusion bimodules. These modules are assigned dimension functions, also denoted by d , as follows: for each basis element m , set $d(m) = \sqrt{d(\text{End}(M))}$, where M is the corresponding simple object in the module category. Then d is multiplicative for the module multiplication as well, and does not depend on whether one takes the left or right internal end in invertible bimodule categories.

The above processes assigns to every fusion category a fusion ring, to every module category a fusion module, and to any bimodule category a fusion bimodule. Given an abstract fusion ring, it is natural to wonder whether it comes from a fusion category. If it does, we will say that \mathcal{C} realizes the fusion ring R , and similarly we will talk about a module or bimodule category realizing a fusion module or bimodule.

If A is an algebra in \mathcal{C} , then the principal graph of A is the induction-restriction graph between \mathcal{C} and $A\text{-mod}$, while the dual principal graph is the induction-restriction graph between $A\text{-mod}$ and $A\text{-mod-}A$. The principal graphs record only part of the fusion bimodule structure, namely the fusion rules for tensoring with the basic bimodules ${}_1A_A$ and ${}_AA_1$. The principal and dual graphs of a finite-depth subfactor $N \subset M$ are the graphs of the algebras $\bar{\iota}$ and $\bar{\iota}$ in the principal and dual even parts, where $\iota : N \rightarrow M$ is the inclusion map.

We refer to [10] for the definitions of fusion modules and bimodules, and multiplicative compatibility for triples of fusion modules and bimodules; however we briefly summarize the basic idea.

Let ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ and ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$ be invertible bimodule categories over fusion categories, and let ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}} \cong {}_{\mathcal{C}}\mathcal{L}_{\mathcal{E}}$. Let ${}_CM_D$, ${}_DN_E$, and ${}_CL_E$ be the corresponding fusion bimodules over fusion rings. Then the equivalence of bimodule categories induces a bimodule homomorphism from ${}_CM_D \otimes_D {}_DN_E$ to ${}_CL_E$ such that if m and n are basis elements in ${}_CM_D$ and ${}_DN_E$ respectively, then the image of $m \otimes n$ is a non-negative combination of basis elements of ${}_CL_E$; moreover, this homomorphism preserves Frobenius–Perron dimension and multiplication by duals in the fusion bimodules. The existence of such a homomorphism places strong combinatorial constraints on the triple $({}_CM_D, {}_DN_E, {}_CL_E)$, which can be checked tediously by hand or quickly with a computer.

Remark 2.17. As mentioned above, in general one needs to be a bit careful about the associativity of compositions of bimodule categories, and this is true even at the level of fusion bimodules. For AH, the vanishing of π_2 guarantees that the composition of bimodules is well-defined, and also that these compositions are associative at the fusion bimodule level. (Whether one can give compatible associators is a more subtle question which involves π_3 as well.) Nonetheless, we will never need to use this uniqueness or associativity.

2.5. The Asaeda–Haagerup fusion categories. The Asaeda–Haagerup subfactor, constructed in [2], is a finite depth subfactor with index $\frac{5+\sqrt{17}}{2}$ and principal and dual graph pair



A subfactor with index $\frac{7+\sqrt{17}}{2}$ and principal and dual graph pair



was constructed in [1], and a third subfactor with index $\frac{9+\sqrt{17}}{2}$ and principal and dual graph pair



was constructed in [10].

Let \mathcal{AH}_2 , \mathcal{AH}_3 , and \mathcal{AH}_1 be the principal even parts of these three subfactors, respectively (\mathcal{AH}_1 is also the dual even part of all three subfactors). The fusion ring of \mathcal{AH}_i is denoted AH_i and the structure constants for these fusion rings are given in the Appendix. In [10] all possible fusion modules and fusion bimodules over these AH_i were computed. Each of these fusion modules can be described by a small list of small matrices with small integer entries (here small means single digit) where each matrix describes the combinatorics of tensoring with one of the simple objects.

Our main result from that paper was a description of all Morita equivalences between any two of the \mathcal{AH}_i .

Theorem 2.18. *There are exactly four invertible bimodule categories over each not-necessarily-distinct pair \mathcal{AH}_i – \mathcal{AH}_j , up to equivalence.*

Furthermore, the Brauer–Picard group of each \mathcal{AH}_i is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The fusion rules for all 36 bimodule categories between the various \mathcal{AH}_i – \mathcal{AH}_j were given in an online appendix to [10]. In addition, the combinatorial possibilities for any other fusion categories in the same Morita equivalence class were severely constrained.

Theorem 2.19 ([10, Theorem 6.10]). *Let $\mathcal{C}^{(k)}$ be a fusion category which is Morita equivalent to the \mathcal{AH}_i (for $i = 1, 2, 3$), but not equivalent to any of them. Then there is a triple of fusion bimodules $(K_{AH_1}^{(k)}, L_{AH_2}^{(k)}, M_{AH_3}^{(k)})$ such that every $\mathcal{C}^{(k)}$ – \mathcal{AH}_1 Morita equivalence realizes $K_{AH_1}^{(k)}$, every $\mathcal{C}^{(k)}$ – \mathcal{AH}_2 Morita equivalence realizes $L_{AH_2}^{(k)}$, and every $\mathcal{C}^{(k)}$ – \mathcal{AH}_3 Morita equivalence realizes $M_{AH_3}^{(k)}$.*

Moreover, there are exactly four possibilities for the triple $(K_{AH_1}^{(k)}, L_{AH_2}^{(k)}, M_{AH_3}^{(k)})$.

Since we will need the details of the four possibilities for $(K_{AH_1}^{(k)}, L_{AH_2}^{(k)}, M_{AH_3}^{(k)})$, we include them in the Appendix for the convenience of the reader. We will refer to these cases using $k = 4, 5, 6, 7$. We will see that cases (4), (5), and (6) have a somewhat similar flavor to each other, while case (7) is quite different. These cases correspond to cases (c), (a), (b), and (d) respectively in [10, Theorem 6.10].

Note that the situation described by Theorem 2.19 is very different than what we see for \mathcal{AH}_{1-3} , for which the fusion bimodule of each \mathcal{AH}_i – \mathcal{AH}_j invertible bimodule category are distinct from each other either as a left AH_i fusion module or as a right AH_j fusion module.

3. Classification of the Morita equivalence class of the Asaeda–Haagerup fusion categories

The main result of this section is the following.

Theorem 3.1. *One of the following two claims is true:*

- *There are no fusion categories realizing any of cases (4), (5), or (6).*
- *There is a unique fusion category realizing each of cases (4), (5), and (6), but there are no fusion categories realizing case (7).*

Remark 3.2. In Section 4 we will see that the latter possibility is the correct one by realizing case (4).

Our first goal will be to constrain the possible fusion rings for the dual fusion categories of the fusion modules $(K_{AH_1}^{(k)}, L_{AH_2}^{(k)}, M_{AH_3}^{(k)})$.

3.1. Finding the possible dual fusion modules of a fusion module. Given a module category over a fusion category $\mathcal{C}\mathcal{M}$, we would like to be able to compute the dual fusion category $\mathcal{D} = (\mathcal{C}\mathcal{M})^*$, and in particular its fusion ring. This former computation can be difficult even when the module category is well understood. However, just the fusion module structure of $\mathcal{C}\mathcal{M}$ can provide strong combinatorial constraints on the fusion rules of \mathcal{D} , and this is sometimes sufficient to calculate the fusion ring of \mathcal{D} and the dual fusion module of $\mathcal{M}_{\mathcal{D}}$, or at least to narrow it down to a few choices.

The idea is that Frobenius reciprocity relates the combinatorics of the two even parts of a subfactor. Suppose that ι is the inclusion $N \rightarrow M$ and that ψ and χ are other M – N sectors. By Frobenius reciprocity, we have that $(\iota\bar{\psi}, \iota\bar{\chi}) = (\bar{\iota}, \bar{\chi}\psi)$. But $(\iota\bar{\psi}, \iota\bar{\chi})$ counts the number of length-2 paths between ψ and χ in the dual principal graph, while $(\bar{\iota}, \bar{\chi}\psi)$ is asking about fusing two odd bimodules into the principal even part. Thus we can read off the number of length-2 paths on the dual principal graph just from understanding the principal half very well. Just knowing the odd vertices and the number of length-2 paths between each pair of odd vertices is often enough to determine the even vertices as well. More generally, the following lemma is an immediate consequence of Frobenius reciprocity.

Lemma 3.3. *Let (K, T) be a fusion bimodule over the fusion rings (A, R) and (B, S) (where the second argument in each pair refers to the basis). For fixed $\kappa \in T$, define the matrices M^κ and N^κ as follows:*

$$M_{ij}^\kappa = \sum_{\xi \in R} (\xi\kappa, \kappa)(\xi\mu_i, \mu_j), \quad \mu_i, \mu_j \in T,$$

and

$$N_{ij}^\kappa = (\mu_i, \kappa\eta_j), \quad \eta_j \in S, \mu_i \in T.$$

Then $N^\kappa(N^\kappa)^t = M$.

Proof. We have

$$\begin{aligned} (N^\kappa(N^\kappa)^t)_{ij} &= \sum_k (\mu_i, \kappa\eta_k)(\mu_j, \kappa\eta_k) \\ &= (\bar{\kappa}\mu_i, \bar{\kappa}\mu_j) \\ &= (\kappa\bar{\kappa}, \mu_j\bar{\mu}_i) \\ &= \sum_{\xi \in R} (\xi\kappa, \kappa)(\xi\mu_i, \mu_j) = M_{ij}^\kappa. \end{aligned} \quad \square$$

The matrix M^κ in Lemma 3.3 is part of the data of the left fusion module ${}_A K$, while the matrix N is the fusion matrix of κ with respect to the right action of B . Thus given ${}_A K$, we can compute all possibilities for the right fusion matrix of κ in a fusion bimodule ${}_A K_B$ whose

left fusion module is ${}_A K$ if we can find all decompositions of M^κ into NN^t for non-negative integer matrices N . Any choice of N determines the dimensions of the basis elements in B , and we may immediately discard any choices which do not allow for a basis element of dimension 1 (the identity) as well as any choices for which there is some other $\kappa' \in T$ which does not admit a right fusion matrix corresponding to the same dimension values for B .

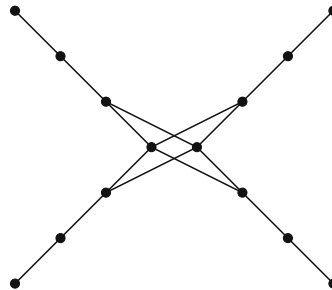
Once we have found all possible right fusion matrices for each basis element of K , we can try to reconstruct all possible fusion bimodules ${}_A K_B$ whose left fusion modules are ${}_A K$, using similar combinatorial searches as in [10].

These can easily be made into computer algorithms all of which run quite quickly since the number of possibilities is small. In fact, all the results of this section were originally checked by computer, but since the calculations are all simple enough we have also checked them all by hand.

3.2. The structure of fusion categories realizing cases (4)–(6). The first step toward proving Theorem 3.1 is to understand the simple objects and fusion rules for any fusion category realizing cases (4)–(6).

Lemma 3.4. *Let \mathcal{C} be a fusion category realizing case (4), (5), or (6) of Theorem 2.19. Then:*

- (i) \mathcal{C} has four invertible objects and four simple objects of dimension $4 + \sqrt{17}$.
- (ii) If \mathcal{C} realizes case (4) (resp. case (5)), then there is a simple object in an invertible \mathcal{C} – \mathcal{AH}_2 (resp. \mathcal{C} – \mathcal{AH}_3) bimodule category whose principal graph is



and the adjacency matrix of whose dual graph is the first matrix appearing in the fusion module $L_{AH_2}^{(4)}$ for case (4) (resp. $M_{AH_3}^{(5)}$ for case (5)).

Proof. In each of the three cases, we look for a basis element of small dimension in one of the fusion modules. In case (4), θ_{42}^1 has a 2-supertransitive principal graph (the matrix given in the appendix is the adjacency matrix of this graph). In case (5), θ_{53}^1 also has a 2-supertransitive principal graph. We can apply Lemma 3.3 to these basis elements to obtain the pictured graph; the dimensions of the simple objects in \mathcal{C} are then given by the normalized Frobenius–Perron weights of the graph.

Since cases (4) and (5) are similar, we only consider case (4) in detail. We refer to the table of fusion modules in the Appendix and use the notation from there. For the fusion module $L_{AH_2}^{(4)}$ we consider the object θ_{42}^1 and compute the matrix

$$M_{ij}^{\theta_{42}^1} = \sum_{\phi} (\theta_{42}^1 \phi, \theta_{42}^1) (\theta_{24}^i \phi, \theta_{24}^j)$$

of Lemma 3.3, where ϕ ranges over the simple objects of \mathcal{AH}_2 . Since $(\theta_{42}^1\phi, \theta_{42}^1) = 1$ if ϕ is either 1 or $\alpha\pi$ and $(\theta_{42}^1\phi, \theta_{42}^1) = 0$ otherwise, the rows of $M^{\theta_{42}^1}$ are obtained by adding the rows corresponding to 1 and to $\alpha\pi$ in each submatrix of the fusion module $L_{AH_2}^{(4)}$. Thus we have

$$M^{\theta_{42}^1} = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 4 & 4 \end{pmatrix}.$$

Then from the relation $NN^t = M$ it is easy to see that up to permutations of columns, the left fusion matrix of θ_{42}^1 must be

$$N^{\theta_{42}^1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

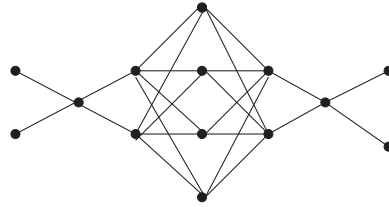
which is the adjacency matrix of the pictured graph.

One can also think of the above calculation as counting paths on principal graphs, which can be an easier way to arrange the calculation if you are doing it by hand. From the fusion matrix for θ_{42}^1 we see that the graph for fusion with θ_{42}^1 has one odd object at depth 1 (θ_{42}^1), two at depth 3 (θ_{24}^5 and θ_{24}^6), and three at depth 5 (θ_{24}^2 , θ_{24}^3 , and θ_{24}^4). Thus the dual graph which we are trying to compute must also have the same number of odd objects at each depth. Now, by Frobenius reciprocity, the matrix M above counts the number of paths of length 2 between these odd vertices. In particular, the vertex at depth one is 2-valent and has exactly one length-2 path to each of the depth-three vertices. Thus, the graph must agree with the pictured graph through depth three. Then the vertices at depth three are 4-valent and since one edge is already accounted for, they must each have three edges connected to depth-four vertices. Since there are four length-2 paths between distinct depth-three vertices (one of which is already accounted for), there must be exactly three vertices at depth 4 each of which is connected to both depth-3 vertices. Thus the graph must agree with the given one through depth 4. Since each of the vertices at depth 5 is 2-valent and only has one length-2 path to each of the depth-3 vertices, each must have a single edge going to a vertex at depth 6. Since these three vertices at depth 5 have no length-2 paths between them, there must be three such vertices at depth 6. Thus the graph must be the one in the theorem.

For case (6), we apply in a similar way Lemma 3.3 to each basis element in the fusion module $M_{AH_3}^{(6)}$. The dimensions listed above are the only choice for which each basis element admits a consistent graph. \square

Remark 3.5. In the rest of this section there are several elementary calculations like the one in the last lemma. Since they are all straightforward and similar to the first calculation, we will not go through them in detail.

Remark 3.6. The principal graph of the smallest simple object in an invertible $\mathcal{C}\text{--}\mathcal{AH}_3$ bimodule category realizing case (6) must be



Lemma 3.7. Let \mathcal{C} be a fusion category which admits a module category with an object having the fusion graph pictured in Lemma 3.4. Then there is an order-4 group G such that the basis for the Grothendieck ring of \mathcal{C} is indexed by $\{\alpha_g\}_{g \in G} \cup \{\alpha_g \rho\}_{g \in G}$, satisfying

$$\alpha_g \alpha_h = \alpha_{g+h} \quad \text{and} \quad \rho^2 = 1 + \sum_{g \in G} 2\alpha_g \rho.$$

There are exactly four possibilities for the fusion ring up to isomorphism, corresponding to four choices for the depth preserving duality involution on the basis:

- (1) the trivial involution, where G is the Klein 4-group and $\alpha_g \rho = \rho \alpha_g$,
- (2) the involution which swaps two invertible basis elements, where G is cyclic and $\alpha_g \rho = \rho \alpha_{g^{-1}}$,
- (3) the involution which swaps two non-invertible basis elements, where G is the Klein 4-group, and without loss of generality $\alpha_{(1,0)} \rho = \rho \alpha_{(0,1)}$, and $\alpha_{(0,1)} \rho = \rho \alpha_{(1,0)}$,
- (4) the involution which swaps two invertible basis elements and two non-invertible basis elements, where G is cyclic and $\alpha_g \rho = \rho \alpha_g$.

Proof. It is clear from the principal graph that there are four invertible objects (which must have the fusion rules for some group of size four) and four non-invertible objects. Let ρ be the non-invertible object at depth 2. Tensoring a simple object with ρ counts the number of length-2 paths (from the vertex representing that object to the vertex representing ρ) minus the number of length-0 paths. Since each of the non-invertible objects has a length-2 path to a different invertible object, it follows that all non-invertible objects are of the form $\alpha_g \rho$. Furthermore, ρ^2 has one length-2 path to the identity, three length-2 paths to itself, and two length-2 paths to each of the other non-invertible objects, so $\rho^2 = 1 + \sum_{g \in G} 2\alpha_g \rho$. Finally, it is clear that those are the only four possibilities for depth preserving involutions on the even vertices, and the fusion rules are determined by the duality using that $\overline{\alpha_g \rho} = \rho \overline{\alpha_g}$. \square

In the following arguments we will frequently need to find the list of objects in a fusion category which admit the structure of a division algebra whose category of modules realizes a given fusion module. Since every division algebra is realized as the internal endomorphisms of a simple object in its category of modules, this list can be read off the data of the fusion module.

We give an example of how this works. Suppose we would like to find the list of objects in \mathcal{AH}_3 which admit an algebra structure whose category of modules realizes the fusion module $M_{AH_3}^{(4)}$.

The fusion module $M_{AH_3}^{(4)}$ has three basis elements, with fusion matrices

$$\left(\begin{array}{c|ccc|ccc|ccc} & \theta_{43}^1 \otimes - & & & \theta_{43}^2 \otimes - & & & \theta_{43}^3 \otimes - & & \\ & \theta_{43}^1 & \theta_{43}^2 & \theta_{43}^3 & \theta_{43}^1 & \theta_{43}^2 & \theta_{43}^3 & \theta_{43}^1 & \theta_{43}^2 & \theta_{43}^3 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \beta & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \xi & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 4 \\ \beta\xi & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 4 \\ \xi\beta & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 4 \\ \beta\xi\beta & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 4 \\ \mu & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 2 & 7 \\ \beta\mu & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 2 & 7 \\ \nu & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 6 \end{array} \right),$$

where the columns of each matrix are indexed by the basis of the fusion module and the rows by the basis of the fusion ring of \mathcal{AH}_3 .

To find the underlying object of the algebra $\overline{\theta_{43}^{(i)}}\theta_{43}^{(i)}$, we just look at $\theta_{43}^{(i)}$ column in the fusion matrix for $\theta_{43}^{(i)}$. Thus for any (right) module category \mathcal{M} over \mathcal{AH}_3 which realizes the fusion module M_{AH_3} , there are two algebras with underlying objects $1 + \beta + \mu + \beta\mu$ and one algebra with underlying object $1 + \beta + 4(\xi + \beta\xi + \xi\beta + \beta\xi\beta) + 7(\mu + \beta\mu) + 6\nu$ whose categories of (left) modules are each equivalent to \mathcal{M} . The first two algebras each have dimension $2d + 2$ and the latter algebra has dimension $28d + 4$, where $d = 4 + \sqrt{17}$. The dimensions of the basis elements of $M_{AH_3}^{(1)}$ are thus $\sqrt{2d + 2}$, $\sqrt{2d + 2}$, and $\sqrt{28d + 4}$. The algebras $\theta_{43}^{(i)}\overline{\theta_{43}^{(i)}}$ in the dual category $(\mathcal{M}_{\mathcal{C}})^*$ hence also have dimensions $2d + 2$ or $28d + 4$.

We introduce the following notation. Let γ be a simple algebra in a fusion category \mathcal{C} . Then we denote the category of γ – γ bimodules in \mathcal{C} by \mathcal{C}^γ . Recall that \mathcal{C}^γ is Morita equivalent to \mathcal{C} .

Lemma 3.8. *The following statements hold.*

- (1) *Let γ be an algebra in a fusion category \mathcal{C} , and let δ be the subobject of γ which contains each invertible object of \mathcal{C} with the same multiplicity as γ does and which does not contain any non-invertible objects. Then δ inherits an algebra structure from γ .*
- (2) *Let \mathcal{C} be a fusion category and let γ, γ' be division algebras such that $(\gamma, \gamma') = 1$ (as objects of \mathcal{C}). Then there is a division algebra δ in \mathcal{C}^γ with $d(\delta) = d(\gamma)d(\gamma')$ such that $(\mathcal{C}^\gamma)^\delta$ is equivalent to $\mathcal{C}^{\gamma'}$.*
- (3) *Let \mathcal{C} be a fusion category and let γ be a division algebra in \mathcal{C} with a subalgebra δ . Then there is a division algebra γ' in \mathcal{C}^δ with dimension $d(\gamma') = \frac{d(\gamma)}{d(\delta)}$ such that $(\mathcal{C}^\delta)^{\gamma'}$ is equivalent to \mathcal{C}^γ .*

Proof. (1) This follows from the fact that the tensor product of any two invertible objects is invertible.

(2) Recall that bimodules over division algebras in \mathcal{C} form a rigid 2-category; we will use multiplicative notation for composition of 1-morphisms. Consider the bimodule $\kappa = {}_{\gamma}\gamma'1_1\gamma'_{\gamma}$. By Frobenius reciprocity, $(\kappa, \kappa)_{\gamma\text{-mod-}\gamma'} = (\gamma, \gamma') = 1$, so κ is simple as a γ – γ' bimodule.

Then $\delta = \kappa\bar{\kappa}$ is a simple algebra in \mathcal{C}^γ , and $\bar{\kappa}\kappa$ is a simple algebra in $\mathcal{C}^{\gamma'}$. By multiplicativity of dimension in the 2-category, we have $d(\delta) = d(\gamma)d(\gamma')$.

(3) Let $\kappa = {}_\gamma\gamma_\delta$. Then $\gamma' = \bar{\kappa}\kappa$ is a simple algebra in \mathcal{C}^δ , $\kappa\bar{\kappa}$ is a simple algebra in \mathcal{C}^γ , and from the relation ${}_\gamma\gamma_1 = {}_\gamma\gamma_\delta \cdot {}_\delta\delta_1$ it follows that $d(\gamma) = d(\gamma')d(\delta)$. \square

Remark 3.9. In the case of a fusion category coming from a subfactor, Lemma 3.8 (3) can be interpreted as follows: the algebra γ corresponds to a subfactor $N \subseteq M$, the subalgebra δ corresponds to $N \subseteq P$ for an intermediate subfactor $N \subseteq P \subseteq M$, and then the “quotient” algebra γ' corresponds to $P \subseteq M$.

The goal of this subsection is to determine, for each of cases (4)–(6), which of the four possibilities the fusion ring must be. A priori it might be that there was more than one fusion category realizing one of the cases and that these two fusion categories have different fusion rings, but it turns out that we can determine the fusion rules from the bimodules. First we concentrate on the invertible objects, and then the non-invertible objects.

A fusion category \mathcal{C} all of whose objects are invertible is necessarily equivalent to Vec_G^ω , where G is a finite group and $\omega \in H^3(G, \mathbb{C}^\times)$ determines the associator. If H is a subgroup of G , then $\bigoplus_{h \in H} k_h$ admits an algebra structure if and only if $\omega|_H$ is trivial; the algebra structures are then parametrized by $H^2(H, \mathbb{C}^\times)$. In fact, these are the only division algebras in $\omega \in H^3(G, \mathbb{C}^\times)$ (this can be seen directly, or by applying [20]).

We will need to know a little about the dual categories to the division algebras in Vec_G^ω when G has size 4. For $\text{Vec}_{\mathbb{Z}/4\mathbb{Z}}$ with trivial associator, the dual category over the 2-dimensional simple algebra is $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}^\xi$ with a specific non-trivial associator which we denote ξ , see [19, Example 4.10]. The associator ξ is trivial when restricted to each of the $\mathbb{Z}/2\mathbb{Z}$ factors, but is non-trivial when restricted to the diagonal. In particular, $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}^\xi$ has exactly two non-trivial division algebras both of dimension 2. On the other hand, $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ with trivial associator has three different simple 2-dimensional algebras. By Lemma 3.8 (2), the dual category over any one of the 2-dimensional algebras in $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ has a 4-dimensional simple algebra, and therefore must again be $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ with trivial associator.

If \mathcal{C} is a fusion category, we denote by $\text{Inv}(\mathcal{C})$ the subcategory generated by the invertible objects. Note that by Lemma 3.4, if \mathcal{C} realizes case (4), (5), or (6), then $\text{Inv}(\mathcal{C}) = \text{Vec}_{\mathbb{Z}/4\mathbb{Z}}^\omega$ or $\text{Inv}(\mathcal{C}) = \text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}^\omega$, for some associator ω .

Lemma 3.10. *Let \mathcal{C} be a fusion category realizing case (4).*

- (1) *The associator on $\text{Inv}(\mathcal{C})$ is trivial.*
- (2) *There is a 4-dimensional division algebra γ_4 in \mathcal{C} such that \mathcal{C}^{γ_4} realizes case (6).*
- (3) *For any 2-dimensional division algebra γ_2 in \mathcal{C} , the fusion category \mathcal{C}^{γ_2} realizes case (5).*

Proof. Since \mathcal{C} realizes case (4), it is the dual category to a module category over \mathcal{AH}_3 which realizes the corresponding fusion module $M_{\mathcal{AH}_3}^{(4)}$; therefore it contains a division algebra $\theta_{43}^3 \theta_{43}^3 = \gamma$ of dimension $28d + 4$ as in the discussion preceding Lemma 3.8. All objects in \mathcal{C} have dimension equal to d or to 1, so γ must contain four invertible objects. If ψ is invertible, then since θ_{43}^3 and $\psi\theta_{43}^3$ are simple, $(\psi, \gamma) = (\psi\theta_{43}^3, \theta_{43}^3) \leq 1$. Hence, γ must contain each of the invertible objects exactly once. The direct sum of the invertible objects then forms a subalgebra of γ by Lemma 3.8.

(1) Since there is an algebra structure on the sum of all four invertible objects, the associator on $\text{Inv}(\mathcal{C})$ must be trivial.

(2) Let γ_4 be the subalgebra of γ corresponding to the direct sum of the invertible objects. Then \mathcal{C}^{γ_4} also contains four invertible objects, so it is not equivalent to \mathcal{AH}_1 – \mathcal{AH}_3 and therefore must realize one of the four cases (4)–(7). Moreover, by Lemma 3.8 there must be a division algebra in \mathcal{C}^{γ_4} of dimension $\frac{28d+4}{4} = 7d+1$ whose category of bimodules is equivalent to \mathcal{AH}_3 . But of the $M_{AH_3}^{(i)}$ for $i = 4, 5, 6, 7$, only $M_{AH_3}^{(6)}$ has a basis element with dimension $\sqrt{7d+1}$. Therefore, \mathcal{C}^{γ_4} must realize case (6).

(3) Since \mathcal{C} realizes case (4), it is the dual category to a module category over \mathcal{AH}_2 which realizes the corresponding fusion module $L_{AH_2}^{(4)}$. Therefore, there is a division algebra in \mathcal{C} with dimension $1+d$, and a non-invertible simple object ρ in \mathcal{C} such that $\mathcal{C}^{1+\rho}$ is equivalent to \mathcal{AH}_2 . Let γ_2 be a 2-dimensional division algebra in \mathcal{C} . Since $(\gamma_2, 1+\rho) = 1$, by Lemma 3.8 there is a division algebra of dimension $2(1+d)$ in \mathcal{C}^{γ_2} whose dual category is \mathcal{AH}_2 . But of the four choices for $L_{AH_2}^{(i)}$, only that of case (5) has a basis element with dimension $\sqrt{2(1+d)}$. Therefore \mathcal{C}^{γ_2} must realize case (5). \square

Lemma 3.11. *Let \mathcal{C} be a fusion category realizing case (5).*

- (1) *The associator on $\text{Inv}(\mathcal{C})$ is non-trivial.*
- (2) *There is a 2-dimensional division algebra γ_2 in \mathcal{C} such that \mathcal{C}^{γ_2} realizes case (4).*

Proof. Since \mathcal{C} realizes case (5), it is the dual category to a module category over \mathcal{AH}_2 which realizes the corresponding fusion module $L_{AH_2}^{(5)}$.

(1) Looking at the fusion module $L_{AH_2}^{(5)}$, we see that \mathcal{C} has a division algebra γ with dimension $7d+1$ such that \mathcal{C}^γ is equivalent to \mathcal{AH}_2 . Suppose that the associator on $\text{Inv}(\mathcal{C})$ is trivial. Let γ_4 be a simple algebra of dimension 4 in \mathcal{C} . Since all simple objects in \mathcal{C} have dimension 1 or dimension d , it follows that γ does not contain any non-trivial invertible objects which implies that $(\gamma, \gamma_4) = 1$. By Lemma 3.8, \mathcal{C}^{γ_4} must have a simple algebra δ of dimension $4(7d+1)$ such that $(\mathcal{C}^{\gamma_4})^\delta$ is equivalent to \mathcal{AH}_2 . Also, since \mathcal{C}^{γ_4} has four invertible objects, it must realize one of the cases (4)–(7). But looking at the four possibilities for $L_{AH_2}^{(i)}$, we find that none of them contain a basis element with dimension $\sqrt{4(7d+1)}$. Therefore, γ_4 cannot exist and the associator on $\text{Inv}(\mathcal{C})$ must be non-trivial.

(2) Again looking at $L_{AH_2}^{(5)}$, we see that there is a division algebra γ in \mathcal{C} of dimension $2(d+1)$ such that \mathcal{C}^γ is equivalent to \mathcal{AH}_2 . Then γ must contain two invertible objects, and therefore contains a 2-dimensional subalgebra γ_2 . Then by Lemma 3.8, \mathcal{C}^{γ_2} has a division algebra γ' of dimension $d+1$ such that $(\mathcal{C}^{\gamma_2})^{\gamma'}$ is equivalent to \mathcal{AH}_2 . Therefore, \mathcal{C}^{γ_2} realizes case (4). \square

Lemma 3.12. *Let \mathcal{C} be a fusion category realizing case (6).*

- (i) *The associator on $\text{Inv}(\mathcal{C})$ is trivial.*
- (ii) *There is a 4-dimensional division algebra γ_4 in \mathcal{C} such that \mathcal{C}^{γ_4} realizes case (4).*

Proof. The proof is similar to the previous lemmas. (Namely, look at the fusion module $L_{AH_2}^{(6)}$ to find a division algebra in \mathcal{C} of dimension $4(d+1)$ which has a 4-dimensional subalgebra γ_4 . Then there is a division algebra γ' in \mathcal{C}^{γ_4} of dimension $d+1$ such that $\mathcal{C}^{\gamma'}$ is equivalent to \mathcal{AH}_2 .) \square

Lemma 3.13. *The following statements hold.*

- (1) *If \mathcal{C} is a fusion category realizing case (4) or case (6), then $\text{Inv}(\mathcal{C})$ is equivalent to $\text{Vec}_{\mathbb{Z}/4\mathbb{Z}}$ with trivial associator.*
- (2) *If \mathcal{C} is a fusion category realizing case (5), then $\text{Inv}(\mathcal{C})$ is equivalent to $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}^{\xi}$.*

Proof. (1) First assume that \mathcal{C} realizes case (4). By Lemma 3.10, $\text{Inv}(\mathcal{C})$ has a trivial associator, but we need to determine whether the group is cyclic or not. Furthermore, we know that there is a 2-dimensional algebra in \mathcal{C} which gives a Morita equivalence between the category $\text{Inv}(\mathcal{C})$ and $\text{Inv}(\mathcal{D})$, where \mathcal{D} is a fusion category realizing case (5). But $\text{Inv}(\mathcal{D})$ has non-trivial associator by Lemma 3.11. In $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ any 2-dimensional algebra has dual category $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ with trivial associator, thus $\text{Inv}(\mathcal{C})$ must be equivalent to $\text{Vec}_{\mathbb{Z}/4\mathbb{Z}}$.

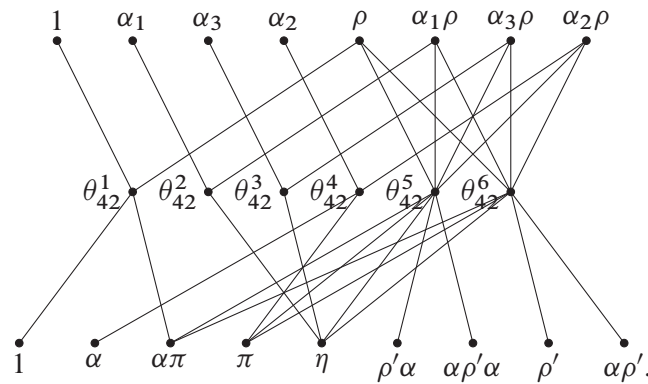
Now assume \mathcal{C} realizes case (6). Then by Lemma 3.12, $\text{Inv}(\mathcal{C})$ is Morita equivalent to $\text{Inv}(\mathcal{D})$, where \mathcal{D} is a fusion category realizing case (4) and the Morita equivalence comes from a 4-dimensional division algebra. Therefore $\text{Inv}(\mathcal{C})$ is also equivalent to $\text{Vec}_{\mathbb{Z}/4\mathbb{Z}}$.

(2) By Lemma 3.11, $\text{Inv}(\mathcal{C})$ is Morita equivalent to $\text{Inv}(\mathcal{D})$, where \mathcal{D} realizes case (4) and the Morita equivalence comes from a 2-dimensional algebra. Since $\text{Inv}(\mathcal{C})$ is equivalent to $\text{Vec}_{\mathbb{Z}/4\mathbb{Z}}$ with trivial associator, $\text{Inv}(\mathcal{C})$ is equivalent to $\text{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}^{\xi}$ by [19]. \square

Now we turn our attention to the non-invertible objects.

Lemma 3.14. *Let \mathcal{C} be a fusion category realizing case (4). Then all non-invertible simple objects in \mathcal{C} are self-dual.*

Proof. This can be deduced by looking at the principal/dual graph pair for θ_{24}^1 in $L_{AH_2}^{(4)}$, where the top row corresponds to objects in \mathcal{C} and the bottom row corresponds to objects in \mathcal{AH}_2 (we use the label ρ' in \mathcal{AH}_2 to distinguish the object from the similarly named ρ in \mathcal{C}):



By the usual associativity condition (cf. [17, Section 3.1]), for any two vertices θ^i, θ^j on the middle row, the number of paths between θ^i and θ^j formed by taking an edge from θ^i to the top row, switching to the dual of that vertex in the top row, and then taking an edge down to θ^j , must be the same as the number of similar paths taken through the bottom row.

Since the two vertices connected to θ_{42}^2 in the bottom row (α and η) both correspond to selfdual objects in \mathcal{AH}_2 , there are exactly two paths from θ_{42}^2 to itself through the bottom graph. Thus there must also be two paths from θ_{42}^2 to itself through the top graph. The only way

this can happen is if $\alpha_2\rho$ is self-dual. Similarly, looking at θ_{42}^1 , we see that ρ must be self-dual. There is exactly one path through the bottom graph between θ_{42}^2 and θ_{42}^3 , so there must also be only one graph between them in the top graph. Since α_1 and α_3 are dual to each other, there is a path going through those vertices. Thus there cannot be a path going through $\alpha_1\rho$ and $\alpha_3\rho$, and so those vertices can not be dual to each other and so must be self-dual. \square

By Lemma 3.7, this completely determines the fusion ring of any fusion category realizing case (4); we call this fusion ring AH_4 . Let AH_5 be the fusion ring satisfying the fusion rules of Lemma 3.7 for $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with all non-invertible basis elements self-dual.

Lemma 3.15. *The following statements hold.*

- (1) *The fusion ring of any fusion category realizing cases (4) or (6) is AH_4 .*
- (2) *The fusion ring of any fusion category realizing case (5) is AH_5 .*

Proof. For case (5), a similar argument as in Lemma 3.14 works to show that all the non-invertible objects are self-dual. For case (6) it is a bit more complicated but it can be checked that if two of the non-invertible objects were dual to each other, the fusion ring would not admit a fusion bimodule with AH_3 realizing the fusion module $M_{AH_3}^{(6)}$. \square

Remark 3.16. Using a computer we can compute all the fusion modules over AH_4 , of which there are 28 up to isomorphism. Similarly, there are 126 fusion modules over AH_5 . We initially found many of the results below by doing computer case analysis for all these fusion modules similar to the arguments in [10]. In order to make this paper widely accessible, we have not used these computer arguments below and instead given alternate by-hand proofs that do not split up into so many cases.

3.3. Uniqueness for cases (4)–(6).

Theorem 3.17. *If there exists a fusion category realizing any of cases (4)–(6), then there exists a unique fusion category realizing each of these three cases.*

Proof. If case (4) can be realized, then so can cases (5) and (6) as bimodules over algebras of dimension 2 and 4 respectively. Similarly, if case (5) or (6) can be realized, then so can (4) as bimodules over algebras of dimension 2 and 4 respectively. So we need only show that there is at most one fusion category realizing each case.

First we consider case (4). Let \mathcal{C}_1 and \mathcal{C}_2 be fusion categories realizing case (4), and let $\mathcal{C}_1\mathcal{K}\mathcal{C}_2$ be a Morita equivalence between them. Then there exist Morita equivalences $\mathcal{C}_1\mathcal{K}_1\mathcal{AH}_2$ and $\mathcal{AH}_2\mathcal{K}_2\mathcal{C}_2$ and a bimodule equivalence $\mathcal{C}_1\mathcal{K}_1\mathcal{AH}_2 \boxtimes_{\mathcal{AH}_2} \mathcal{AH}_2\mathcal{K}_2\mathcal{C}_2 \rightarrow \mathcal{C}_1\mathcal{K}\mathcal{C}_2$. Since any bimodule between \mathcal{AH}_2 and \mathcal{C}_1 or \mathcal{C}_2 corresponds to the fusion module $L_{AH_2}^{(4)}$, there are simple objects $\kappa_1 \in \mathcal{K}_1$ and $\kappa_2 \in \mathcal{K}_2$ corresponding to the basis element θ_{42}^1 such that $\bar{\kappa}_1\kappa_1 = 1 + \rho = \kappa_2\bar{\kappa}_2$ in \mathcal{AH}_2 . By Frobenius reciprocity, $\kappa_1\kappa_2$ in \mathcal{K} also has two simple summands and dimension $1 + d$. Since each simple object in \mathcal{K} has dimension equal to $\sqrt{i + jd}$ for some non-negative integers $i \leq 4$ and $j \leq 32$, this is only possible if the dimensions of the two simple summands are 1 and d , or 2 and $d - 1$. In the latter case, the internal endomorphisms of the 2-dimensional object, would give a 4-dimensional division algebra in \mathcal{C}_1 whose

dual was \mathcal{C}_2 . But this is impossible because such a dual category would have to realize case (6) and not case (4). Thus, \mathcal{K} has an object of dimension one whose internal endomorphisms must be the trivial algebra. Hence, \mathcal{C}_2 is equivalent to the category of bimodules over the trivial algebra in \mathcal{C}_1 and thus is equivalent to \mathcal{C}_1 .

It follows that cases (5) and (6) are also each realized by at most one fusion category. Any fusion category \mathcal{C} which realizes case (6) has a 4-dimensional division algebra γ_4 such that \mathcal{C}^{γ_4} realizes case (4). In particular, \mathcal{C} must be the category of bimodules for a 4-dimensional algebra in the unique category realizing case (4) and thus is itself unique. Similarly, any fusion category \mathcal{C} realizing case (5) has a 2-dimensional algebra γ_2 such that \mathcal{C}^{γ_2} realizes case (4), so it also must be the category of bimodules for the unique 2-dimensional algebra in the unique fusion category realizing case (4). \square

We will call the three (thus far hypothetical, but unique if they exist) fusion categories which realize these three cases \mathcal{AH}_4 , \mathcal{AH}_5 , and \mathcal{AH}_6 .

Theorem 3.18. *If \mathcal{AH}_{4-6} exist, then every Morita autoequivalence of each of these fusion categories is realized by an outer automorphism.*

Proof. We first consider \mathcal{AH}_4 , which has fusion ring AH_4 . As in the proof of the previous lemma, any Morita autoequivalence must come from the trivial algebra, and is therefore realized by an outer automorphism.

Next we consider \mathcal{AH}_5 . There is a Morita equivalence between \mathcal{AH}_4 and \mathcal{AH}_5 which is implemented by a 2-dimensional algebra. Since every Morita autoequivalence of \mathcal{AH}_4 is realized by an outer automorphism, in fact every \mathcal{AH}_4 – \mathcal{AH}_5 autoequivalence is implemented by a 2-dimensional algebra. Again as in the proof of the previous lemma, by Frobenius reciprocity this means that every Morita autoequivalence of \mathcal{AH}_5 contains an object with dimension 2 and two simple summands. This means the summands each have dimension 1, and the autoequivalence comes from the trivial module category, and is therefore realized by an outer automorphism. For \mathcal{AH}_6 we use the same argument, starting with the \mathcal{AH}_5 – \mathcal{AH}_6 Morita equivalence coming from a 2-dimensional algebra. \square

Corollary 3.19. *If \mathcal{AH}_{4-6} exist, then $\text{Out}(\mathcal{AH}_i) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for $i = 4, 5, 6$.*

3.4. Non-existence for case (7). In this subsection we show that if \mathcal{AH}_{4-6} exist (as we will prove in Section 4), then case (7) cannot be realized.

Lemma 3.20. *Let \mathcal{C} be a fusion category realizing case (7). Then every invertible \mathcal{C} – \mathcal{AH}_5 bimodule category has a simple object of dimension $\sqrt{1+5d}$.*

Proof. Every invertible \mathcal{C} – \mathcal{AH}_5 bimodule category \mathcal{K} can be expressed as a relative tensor product $\mathcal{K} = \mathcal{K}_1 \boxtimes_{\mathcal{AH}_3} \mathcal{K}_2$ of an invertible \mathcal{C} – \mathcal{AH}_3 bimodule category \mathcal{K}_1 and an invertible \mathcal{AH}_3 – \mathcal{AH}_5 bimodule category \mathcal{K}_2 . Since \mathcal{C} realizes case (7) and \mathcal{AH}_5 realizes case (5), there are simple objects $\theta_{73}^1 \in \mathcal{K}_1$ and $\theta_{53}^1 \in \mathcal{K}_2$ such that

$$\overline{\theta_{73}^1} \theta_{73}^1 = 1 + \beta\xi + \xi\beta + \mu \quad \text{and} \quad \overline{\theta_{53}^1} \theta_{53}^1 = 1 + \mu$$

(both objects in \mathcal{AH}_3). Therefore by Frobenius reciprocity, the object $\theta_{73}^1 \overline{\theta_{53}^1} \in \mathcal{K}$ has two

simple summands, and its dimension is $\sqrt{(2+2d)(1+d)} = \sqrt{2}(1+d)$. Since every object in \mathcal{K} must have dimension $\sqrt{i+jd}$ for non-negative integers i, j , and neither summand can have integer dimension, the only possibility is that each summand has dimension

$$\sqrt{1+5d} = \frac{\sqrt{2}(1+d)}{2}. \quad \square$$

In the following two lemmas we need to discuss properties of \mathcal{AH}_5 . We label the simple objects by α_g and $\alpha_g\rho$, $g \in G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where ρ is a non-invertible simple object.

Lemma 3.21. *The object $1 + \alpha_g\rho$ in \mathcal{AH}_5 admits two algebra structures whose dual categories are \mathcal{AH}_3 for each $g \in G$.*

Proof. Since \mathcal{AH}_5 satisfies case (5), each of the four \mathcal{AH}_5 – \mathcal{AH}_3 Morita equivalences is implemented by an algebra of dimension $1+d$ in \mathcal{AH}_5 . Let $1 + \alpha_{g_i}\rho$, $1 \leq i \leq 4$, be algebras corresponding to the four Morita equivalences, and suppose that $g_i = g_j$ for some $i \neq j$. Then as in the proof of Theorem 3.17, we can compose the two bimodule objects corresponding to the algebras $1 + \alpha_{g_i}\rho$ and $1 + \alpha_{g_j}\rho$ to obtain an object in an \mathcal{AH}_3 –Morita autoequivalence which has dimension $1+d$ and two simple summands. Moreover, the Morita autoequivalence must be non-trivial since $i \neq j$. However, there are no such objects in the non-trivial Morita autoequivalences of \mathcal{AH}_3 . Therefore all of the g_i are distinct, and each $1 + \alpha_{g_i}\rho$ admits an algebra structure whose dual category is \mathcal{AH}_3 . Since the fusion module $M_{\mathcal{AH}_3}^{(5)}$ contains four distinct basis elements of dimension $\sqrt{1+d}$, which form two equivalence classes under the action of the group of invertible objects of \mathcal{AH}_3 , there are in fact two such algebras for each $g \in G$. \square

Corollary 3.22. *The outer automorphism group of \mathcal{AH}_5 acts transitively on the non-invertible simple objects.*

Remark 3.23. The situation is quite different for \mathcal{AH}_4 . If we let β_h , $h \in H = \mathbb{Z}/4\mathbb{Z}$, be the invertible objects of \mathcal{AH}_4 and let ξ be a non-invertible simple object, then two of the \mathcal{AH}_4 – \mathcal{AH}_2 Morita equivalences correspond to pairs of algebra structures on $1 + \xi$ and $1 + \beta_2\xi$ and the other two Morita equivalences correspond to pairs of algebra structures on $1 + \beta_1\xi$ and $1 + \beta_3\xi$. The argument in the proof of the previous lemma fails here since, unlike for \mathcal{AH}_3 , there is a non-trivial Morita autoequivalence of \mathcal{AH}_2 with an object of dimension $1+d$ and two simple summands. Specifically, there is a Morita autoequivalence of \mathcal{AH}_2 with simple objects of dimension $\frac{d+1}{2}, \frac{d+1}{2}, \frac{d+1}{2}, \frac{d+1}{2}, d-1, d-1, d+1$ (see [10]).

Theorem 3.24. *If \mathcal{AH}_{4-6} exist, there do not exist any fusion categories in the Morita equivalence class of the \mathcal{AH}_i realizing case (7).*

Proof. Let \mathcal{C} be a fusion category realizing case (7). By Lemma 3.20, there exists an object κ of dimension $\sqrt{1+5d}$ in an invertible \mathcal{C} – \mathcal{AH}_5 bimodule category. Then $\bar{\kappa}\kappa$ contains five summands with dimension d , so at least one simple summand has multiplicity at least 2. Since the outer automorphism group acts transitively on the non-invertible simple objects, we may assume without loss of generality that summand is ρ , and we have

$$(\bar{\kappa}\kappa, \rho) \geq 2.$$

Let θ_{53}^1 be a simple object in an invertible \mathcal{AH}_5 – \mathcal{AH}_3 bimodule category whose internal End is $1 + \rho$. Then

$$(\kappa \boxtimes \theta_{53}^1, \kappa \boxtimes \theta_{53}^1) = (\bar{\kappa}\kappa, \theta_{53}^1 \overline{\theta_{53}^1}) \geq 3$$

and

$$d(\kappa \boxtimes \lambda) = \sqrt{(1+d)(1+5d)} = \sqrt{46d+6}.$$

However, the dimensions of the basis elements in $N_{\mathcal{AH}_3}$ are

$$d(\theta_{73}^1) = d(\theta_{73}^2) = \sqrt{2d+2}, \quad d(\theta_{73}^3) = \sqrt{28d+4}.$$

Therefore there cannot be an object \mathcal{C} – \mathcal{AH}_3 bimodule category with at least three simple summands and dimension $\sqrt{46d+6} = d(\theta_{73}^1) + d(\theta_{73}^3)$. \square

4. Existence of \mathcal{AH}_{4-6}

In the previous section we established that if any of cases (4)–(6) of Theorem 2.19 occur, then there are exactly three additional fusion categories in the Morita equivalence class of \mathcal{AH}_{1-3} , which we called \mathcal{AH}_{4-6} . In this section we will show that \mathcal{AH}_{4-6} exist by explicitly constructing \mathcal{AH}_4 as a category of finite-index endomorphisms of a Type III factor M . By Lemmas 3.7, 3.10, and 3.14, \mathcal{AH}_4 must contain a copy of $\text{Vec}_{\mathbb{Z}/4\mathbb{Z}}$, with simple objects $\{\alpha_g\}_{g \in \mathbb{Z}/4\mathbb{Z}}$, along with a self-dual simple object ρ , satisfying the fusion rules

$$\alpha_g \rho \cong \rho \alpha_{-g} \quad \text{for all } g \in \mathbb{Z}/4\mathbb{Z}, \quad \text{and} \quad \rho^2 \cong 1 + 2 \sum_{g \in \mathbb{Z}/4\mathbb{Z}} \alpha_g \rho.$$

A general theory of such categories for arbitrary finite abelian groups, but with the multiplicities of $\alpha_g \rho$ in ρ^2 all equal to 1 instead of 2, has been developed by the second-named author in [11, 13, 14]. He determined a complete numerical invariant of such categories given by solutions of certain polynomial equations determined by the group G ; these solutions give structure constants for endomorphisms $\{\alpha_g\}_{g \in G}$ and ρ of the Cuntz algebra $\mathcal{O}_{|G|+1}$, which can then be extended to endomorphisms of a Type III factor. In this construction one Cuntz algebra generator belongs to each intertwiner space (α_g, ρ^2) and one generator belongs to $(1, \rho^2)$.

We could try to perform a similar construction for \mathcal{AH}_4 by realizing ρ as an endomorphism of $\mathcal{O}_{2|G|+1} = \mathcal{O}_9$, again with one generator belonging to $(1, \rho^2)$, but this time with two generators belonging to each (α_g, ρ^2) . However, because the intertwiner spaces are two-dimensional, it is difficult to write down equations for the structure constants of ρ . We therefore take a different approach, and construct instead an equivariantization of \mathcal{AH}_4 , which is technically easier to deal with. We then recover \mathcal{AH}_4 by de-equivariantizing.

4.1. Equivariantization and the orbifold construction. We recall the notion of equivariantization of a fusion category by the action of a finite group of automorphisms. For simplicity we define equivariantization for strict categories, which include categories of endomorphisms (though as with all constructions for monoidal categories, strictness is not really important). Let \mathcal{C} be a strict fusion category, and let G be a finite group acting by automorphisms $\{\phi_g\}_{g \in G}$ on \mathcal{C} . A G -equivariant object of \mathcal{C} is an object $\xi \in \mathcal{C}$, together with isomorphisms $\{u_g : \phi_g(\xi) \rightarrow \xi\}_{g \in G}$, such that

$$u_g \phi_g(u_h) = u_{gh}.$$

Morphisms and tensor product for G -equivariant objects can be defined in a natural way, and the G -equivariant objects form a fusion category, called the G -equivariantization of \mathcal{C} .

Suppose \mathcal{C} is a fusion category of finite-index endomorphisms of a Type III factor M , and suppose α is an order two automorphism of M which commutes with a set of representative endomorphisms of the simple objects of \mathcal{C} . Since \mathcal{C} is equivalent to its full subcategory of endomorphisms which commute with α , we assume without loss of generality that α commutes with all endomorphisms in \mathcal{C} . Then α induces a $\mathbb{Z}/2\mathbb{Z}$ -action on \mathcal{C} , where the action of α fixes objects and takes each morphism $u \in (\xi, \sigma)$ to $\alpha(u) \in (\xi, \sigma)$. The equivariant objects with respect to this action are of the form (ξ, u) , with $u \in (\xi, \xi)$ satisfying $u\alpha(u) = 1$. In particular, for each simple object $\xi \in \mathcal{C}$, there are two inequivalent objects in the $\mathbb{Z}/2\mathbb{Z}$ -equivariantization, corresponding to $u = \pm 1$.

Now suppose that α is not in the same sector as any object of \mathcal{C} . Then the $\mathbb{Z}/2\mathbb{Z}$ -equivariantization of \mathcal{C} induced by α can be realized as a category of endomorphisms on a larger factor via an orbifold construction, as follows. Let M_1 be the crossed product $M \rtimes_{\alpha} \mathbb{Z}/2\mathbb{Z}$, which is the von Neumann algebra generated by M and a self-adjoint unitary λ satisfying

$$\lambda x \lambda = \alpha(x) \quad \text{for all } x \in M.$$

Then every equivariant object (ξ, u) can be extended to an endomorphism $\tilde{\xi}$ on M_1 by setting $\tilde{\xi}(\lambda) = u\lambda$, and the morphisms between pairs of such extended endomorphisms $\tilde{\xi}_1$ and $\tilde{\xi}_2$ in $\text{End}_0(M_1)$ are precisely the equivariant morphisms between ξ_1 and ξ_2 in $\text{End}_0(M)$.

4.2. $\mathbb{Z}/2\mathbb{Z}$ -equivariantization of \mathcal{AH}_4 . This subsection is purely motivational and explains why if \mathcal{AH}_4 exists, then it must have a $\mathbb{Z}/2\mathbb{Z}$ -equivariantization \mathcal{S} which is easier to understand. Logically, this is not necessary, since we will directly construct this \mathcal{S} and use it to construct \mathcal{AH}_4 via the inverse process of de-equivariantization.

The following can be shown with a considerable amount of work.

Lemma 4.1. *Suppose \mathcal{AH}_4 exists. Let $H = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then the $\mathbb{Z}/2\mathbb{Z}$ -equivariantization with respect to the action coming from conjugation by α_2 contains a copy of Vec_H , with simple objects $\{\alpha_g\}_{g \in H}$, along with a simple object ρ , satisfying the fusion rules*

$$\alpha_g \rho \cong \rho \alpha_{-g} \quad \text{for all } g \in H, \quad \text{and} \quad \rho^2 \cong 1 + \sum_{g \in H} \alpha_g \rho.$$

We do not prove this lemma here, since the result is not required for anything that follows and the proof is difficult and detailed. Rather the lemma functions as motivation for constructing \mathcal{AH}_4 by first constructing its $\mathbb{Z}/2\mathbb{Z}$ -equivariantization. However, we note several features of this proof which will motivate some choices in the next section.

Remark 4.2. (1) The proof proceeds by realizing (the hypothetical) \mathcal{AH}_4 as the fusion category generated by automorphisms $\{\alpha_g\}_{g \in \mathbb{Z}/4\mathbb{Z}}$, along with an endomorphism ρ , on a Type III factor M containing a Cuntz algebra \mathcal{O}_9 , with one generator belonging to $(1, \rho^2)$ and two generators belonging to each $(\alpha_g \rho, \rho^2)$. Then proving the lemma amounts to showing that the eigenvalues of the action of α_2 on each $(\alpha_g \rho, \rho^2)$ are $\{1, -1\}$.

(2) Finding the eigenvalues of α_2 required using properties of \mathcal{AH}_4 beyond just the fusion rules and the presence of a $\text{Vec}_{\mathbb{Z}/4\mathbb{Z}}$ subcategory, namely that $1 + \rho$ admits a Q-system, that the dual category with respect to this Q-system contains a non-trivial invertible object, and

that the center of \mathcal{AH}_4 contains no non-trivial invertible objects (and therefore conjugation by α_2 is a non-trivial automorphism). These properties follow from the fact that \mathcal{AH}_4 is assumed to be the dual category of \mathcal{AH}_2 with respect to a Q-system of dimension $1 + d$.

(3) Since α_2 belongs to \mathcal{AH}_4 , we cannot simply use a crossed product by α_2 for the orbifold construction, as that would result in α_2 being identified with the identity automorphism on the Type III factor M . Rather, we need to twist α_2 by an automorphism of M to obtain an automorphism α' which is not in the same sector as any automorphism of \mathcal{AH}_4 , but such that conjugation by α' is equivalent to conjugation by α_2 as an automorphism of \mathcal{AH}_4 .

(4) From the orbifold construction, we can deduce some important structure constants of the $\mathbb{Z}/2\mathbb{Z}$ -equivariantization, which will be useful in the following subsection.

4.3. Constructing \mathcal{S} . By Lemma 4.1, the $\mathbb{Z}/2\mathbb{Z}$ -equivariantization of \mathcal{AH}_4 would have nicer fusion rules than \mathcal{AH}_4 itself. We now directly construct a subfactor \mathcal{S} whose principal even half has these fusion rules.

We recall the following construction from [14], which completely characterizes unitary fusion categories which are tensor generated by Vec_G for a finite abelian group G (which for us is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$) and a simple object ρ , satisfying

$$\alpha_g \rho \cong \rho \alpha_{-g} \quad \text{for all } g \in G, \quad \text{and} \quad \rho^2 \cong 1 + \sum_{g \in G} \alpha_g \rho.$$

Let G be a finite abelian group, and let

$$A : G \times G \times G \rightarrow \mathbb{C}, \quad \epsilon : G \times G \rightarrow \{-1, 1\}, \quad \eta : G \rightarrow \{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$$

be functions. Set $A_g(h, k) = A(g, h, k)$, $\epsilon_g(h) = \epsilon(g, h)$, and $\eta_g = \eta(g)$ for $g, h, k \in G$. Let $n = |G|$ and let

$$d = \frac{n + \sqrt{n^2 + 4}}{2}.$$

Consider the following system of equations:

$$(4.1) \quad \epsilon_{h+k}(g) = \epsilon_h(g)\epsilon_k(g+2h), \quad \epsilon_h(0) = 1,$$

$$(4.2) \quad \eta_{g+2h} = \eta_g, \quad \eta_g^3 = 1,$$

$$(4.3) \quad \sum_{h \in G} A_g(h, 0) = -\frac{\overline{\eta_g}}{d},$$

$$(4.4) \quad \sum_{h \in G} A_g(h - g, k) \overline{A_{g'}(h - g', k)} = \delta_{g, g'} - \frac{\overline{\eta_g} \eta_{g'}}{d} \delta_{k, 0},$$

$$(4.5) \quad A_{g+2h}(p, q) = \epsilon_h(g)\epsilon_h(g+p)\epsilon_h(g+q)\epsilon_h(g+p+q)A_g(p, q),$$

$$(4.6) \quad A_g(h, k) = \overline{A_g(k, h)},$$

$$(4.7) \quad \begin{aligned} A_g(h, k) &= A_g(-k, h-k)\eta_g\epsilon_{-k}(g+h)\epsilon_{-k}(g+k)\epsilon_{-k}(g+h+k) \\ &= A_g(k-h, -h)\overline{\eta_g}\epsilon_{-h}(g+h)\epsilon_{-h}(g+k)\epsilon_{-h}(g+h+k), \end{aligned}$$

$$(4.8) \quad \begin{aligned} A_g(h, k) &= A_{g+h}(h, k)\eta_g\eta_{g+k}\overline{\eta_{g+h}\eta_{g+h+k}}\epsilon_h(g)\epsilon_h(g+k) \\ &= A_{g+k}(h, k)\overline{\eta_g\eta_{g+h}\eta_{g+k}}\eta_{g+h+k}\epsilon_k(g)\epsilon_k(g+h), \end{aligned}$$

and

$$\begin{aligned}
 (4.9) \quad & \sum_{l \in G} A_g(x+y, l) A_{g-p+x}(-x, l+p) A_{g-q+x+y}(-y, l+q) \\
 &= A_g(p+x, q+x+y) A_{g-p}(q+y, p+x+y) \\
 &\quad \times \eta_g \eta_{g+q+x} \eta_{g+p+q+y} \overline{\eta_{g+p} \eta_{g+x+y} \eta_{g+q+x+y}} \\
 &\quad \times \epsilon_p(g-p+x) \epsilon_{p+x}(g-p+q+y) \\
 &\quad \times \epsilon_q(g-q+x+y) \epsilon_{q+y}(g-q+x) \\
 &\quad - \frac{\delta_{x,0} \delta_{y,0}}{d} \eta_g \eta_{g+p} \eta_{g+q}.
 \end{aligned}$$

Theorem 4.3 ([14]). *Suppose A and ϵ satisfy (4.1)–(4.9). Then there is a Type III factor M , an outer action α of G on M , an irreducible finite-index endomorphism ρ of M , and $n+1$ isometries $S \in (\text{id}, \rho^2)$ and $T_g \in (\alpha_g \circ \rho, \rho^2)$ satisfying the Cuntz algebra relations such that*

$$(4.10) \quad \alpha_g \circ \rho = \rho \circ \alpha_{-g} \quad \text{for all } g \in G,$$

$$(4.11) \quad \alpha_g(S) = S, \quad \alpha_g(T_h) = \epsilon_g(h) T_{h+2g} \quad \text{for all } g, h \in G,$$

$$(4.12) \quad \rho(s) = \frac{1}{d} S + \frac{1}{\sqrt{d}} \sum_{g \in G} T_g T_g^*,$$

$$(4.13) \quad \rho(T_g) = \epsilon_g(-g) \left[\eta_{-g} T_{-g} S S^* + \frac{\overline{\eta_{-g}}}{\sqrt{d}} S T_{-g}^* \right.$$

$$(4.14) \quad \left. + \sum_{h,k \in G} A_{-g}(h,k) T_{h-g} T_{h+k-g} T_{k-g}^* \right] \quad \text{for all } g \in G.$$

The sector $[\text{id}] \oplus [\alpha_g \rho]$ admits a Q-system if and only if

$$(4.15) \quad A_g(h, 0) = \delta_{h,0} - \frac{1}{d-1} \quad \text{for all } h \in G.$$

These equations were solved in [14] for all groups of order less than or equal to 5, and solutions have been found for some groups of higher order. For all known solutions for which $[\text{id}] \oplus [\rho]$ admits a Q-system, $[\text{id}] \oplus [\alpha_g \rho]$ admits Q-systems for all $g \in G$. In this case we get some very useful extra relations.

Lemma 4.4 ([14]). *If $[\text{id}] \oplus [\alpha_g \rho]$ admits a Q-system for each $g \in G$, then η_g is identically 1 and ϵ restricted to $G_2 \times G$ is a bicharacter, where G_2 is the subgroup of order-2 elements of G . Moreover, we have*

$$(4.16) \quad |A_g(h, k)|^2 = \delta_{h,0} \delta_{k,0} - \frac{\delta_{h,0} + \delta_{k,0} + \delta_{h,k}}{d-1} + \frac{d}{(d-1)^2},$$

$$\begin{aligned}
 (4.17) \quad & A_g(-l, h) A_g(l, k) \epsilon_l(g-l) \epsilon_l(g+h-l) \\
 & \quad - A_g(h+l, k) A_g(k-l, h) \epsilon_l(g+k-l) \epsilon_l(g+h+k-l) \\
 &= \frac{\delta_{h-k+l,0}}{d-1} \epsilon_l(g+h) - \frac{\delta_{l,0}}{d-1}
 \end{aligned}$$

for all $g, h, k, l \in G$.

The structure of a solution to (4.1)–(4.9) seems to depend in a fundamental way on ϵ , which is determined up to gauge equivalence by its restriction to $G_2 \times G$. From the orbifold construction in the preceding subsection, we can determine that in order for \mathcal{S} to be related to the Asaeda–Haagerup subfactor, the gauge equivalence class of ϵ on $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for the $\mathbb{Z}/2\mathbb{Z}$ -equivariantization of \mathcal{AH}_4 should satisfy

$$\epsilon_{(0,1)}((a,b)) = 1, \quad \epsilon_{(2,0)}((a,b)) = (-1)^b \quad \text{for all } (a,b) \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

We fix ϵ by setting $\epsilon_{(1,0)}((2,1)) = \epsilon_{(1,0)}((3,1)) = -1$ and $\epsilon_{(1,0)}((a,b)) = 1$ otherwise. Together with $\epsilon_{(0,1)}((a,b)) = 1$ and (4.1) this determines all the ϵ .

Then we can try to solve for $A_g(h,k)$ using (4.3)–(4.9) and possibly (4.16)–(4.17). We know of no general technique for solving these equations, but nonetheless were able to find a solution in this particular case.

Theorem 4.5. *There exists a solution to equations (4.1)–(4.9) and (4.15) for the group $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $\epsilon_{(2,0)}((a,b)) = (-1)^b$ and $\epsilon_{(0,1)}((a,b)) = 1$ for all (a,b) .*

Proof. We explicitly construct a solution is as follows. We define constants

$$\begin{aligned} c &= \frac{1}{4}(1 - d + i\sqrt{10d - 2}), \\ f &= \sqrt{\frac{1}{2}(d - 1 - i\sqrt{26d + 2})}, \\ g &= \frac{1}{2}\sqrt{-3d - 1 + i\sqrt{50d + 6}}, \\ h &= \frac{1}{4}(d + 3 - i(\sqrt{2d - 10})). \end{aligned}$$

Define $G \times G$ matrices

$$A = \frac{1}{d-1} \begin{pmatrix} d-2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & c & c & -f & f & -g & -g \\ -1 & \bar{c} & -1 & c & i\sqrt{d} & h & -i\sqrt{d} & \bar{h} \\ -1 & \bar{c} & \bar{c} & -1 & -\bar{f} & -\bar{g} & \bar{g} & -\bar{f} \\ -1 & -\bar{f} & -i\sqrt{d} & -f & -1 & -f & i\sqrt{d} & -\bar{f} \\ -1 & \bar{f} & \bar{h} & -g & -\bar{f} & -1 & g & -\bar{h} \\ -1 & -\bar{g} & i\sqrt{d} & g & -i\sqrt{d} & \bar{g} & -1 & -g \\ -1 & -\bar{g} & h & -f & -f & -h & -\bar{g} & -1 \end{pmatrix},$$

$$B_{(1,0)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix},$$

$$B_{(0,1)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix},$$

$$B_{(1,1)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

We set

$$\begin{aligned} A_{(0,0)}(h, k) &= A(h, k), & A_{(1,0)}(h, k) &= B_{(1,0)}(h, k)A(h, k), \\ A_{(0,1)}(h, k) &= B_{(0,1)}(h, k)A(h, k), & A_{(1,1)}(h, k) &= B_{(1,1)}(h, k)A(h, k) \end{aligned}$$

and use (4.5) to define the remaining $A_g(h, k)$.

We need only check that these numbers satisfy the finitely many polynomial equations (4.1)–(4.9) and (4.15). Although it is possible to check any one of these equations by hand, since there are 40,600 total equations (mostly from equation (4.9)) we checked them by computer. \square

Remark 4.6. The computer calculation in the above theorem has been checked twice independently. Our original program used a mix of C and Mathematica and ran in several hours. A second program was written by Morrison, Penneys, and Peters in Mathematica alone and is more easily human-readable but ran in a couple days. We have included along with the arxiv source to this article an improved version of their code as `CheckingSolution.nb` which is much faster and runs in under three minutes on a 2.4 GHz Intel Core i5. We are grateful to Morrison, Penneys, and Peters for their help improving this code.

Remark 4.7. The initial step in solving (4.1)–(4.9) is determining the cocycle ϵ up to gauge equivalence. In our case we were able to determine what ϵ should be by assuming that the resulting category had an orbifold quotient which is Morita equivalent to the Asaeda–Haagerup categories. Then (4.5)–(4.8) can be used to express the $A_g(h, k)$ in terms of a relatively small number of variables, which can be solved using (4.4) and (4.17) for $g = 0$. Although we were not assuming that $1 + \alpha_g \rho$ admits a Q-system for all g , we were assuming that $1 + \rho$ admits a Q-system, so (4.15) has to hold for $g = 0$ and consequently so does (4.17) for $g = 0$. If one assumes that (4.15) holds for all g (which indeed turns out to be the case), then the equations are easier to solve.

We will call the fusion category corresponding to this solution $\mathcal{AH}^{\mathbb{Z}/2\mathbb{Z}}$ and the subfactor coming from $1 + \rho$ we will call \mathcal{S} .

4.4. De-equivariantization and the orbifold construction. We can now construct \mathcal{AH}_4 by de-equivariantizing $\mathcal{AH}^{\mathbb{Z}/2\mathbb{Z}}$; the de-equivariantization is implemented by another orbifold construction.

Let G be a finite abelian group and let A, ϵ be a solution to (4.1)–(4.9). Let $0 \neq z \in G_2$ be such that $\epsilon_g(\cdot)$ is a character and $\epsilon_z(z) = 1$. Let $P = M \rtimes_{\alpha_z} \mathbb{Z}/2\mathbb{Z}$ be the crossed product, which is the von Neumann algebra generated by M and a unitary λ satisfying $\lambda^2 = 1$ and $\lambda x \lambda = \alpha_z(x)$ for all $x \in M$. We extend α_g and ρ to P by setting $\tilde{\alpha}_g(\lambda) = \epsilon_z(g)\lambda$ and $\tilde{\rho}(\lambda) = \lambda$. Then $\tilde{\alpha}$ is a G -action on P and $\tilde{\alpha}_z = \text{Ad } \lambda$.

Theorem 4.8 ([14]). *The following statements hold.*

- (1) $[\tilde{\rho}]$ is irreducible and self-dual; $[\text{id}] \oplus [\tilde{\rho}]$ admits a Q -system if $[\text{id}] \oplus [\rho]$ does.
- (2) $[\tilde{\alpha}_g] = [\tilde{\alpha}_h]$ if and only if $g - h \in \{0, z\}$.
- (3) $[\tilde{\rho}^2] = [\text{id}] \oplus \bigoplus_{\dot{g} \in G/\{0, z\}} 2[\tilde{\alpha}_{\dot{g}} \tilde{\rho}]$.

We can apply this construction to $\mathcal{AH}^{\mathbb{Z}/2\mathbb{Z}}$.

Theorem 4.9. *There is a Type III factor M with an outer action α of $\mathbb{Z}/4\mathbb{Z}$, an irreducible finite-index endomorphism ρ and isometries $S \in (\text{id}, \rho^2)$ and $T_{(a,b)} \in (\alpha_a \rho, \rho^2)$, where $(a, b) \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, such that*

- (1) $\alpha_g \rho = \rho \alpha_{-g}$ and $[\rho]^2 = [\text{id}] \oplus 2 \sum_{g \in \mathbb{Z}/4\mathbb{Z}} [\alpha_g \rho]$,
- (2) equations (4.11)–(4.13) hold with A and ϵ as in the previous subsection for all α_g , where $g \in \mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. Apply the orbifold construction to $\mathcal{AH}^{\mathbb{Z}/2\mathbb{Z}}$ with $z = (0, 1)$. Note that α_z acts trivially on the original Cuntz algebra, since z has order 2 and ϵ_z is identically 1. After replacing $T_{(a,1)}$ with $T_{(a,1)}\lambda$ for each $a \in \mathbb{Z}/4\mathbb{Z}$, it is easy to check that equations (4.11)–(4.13) still hold. \square

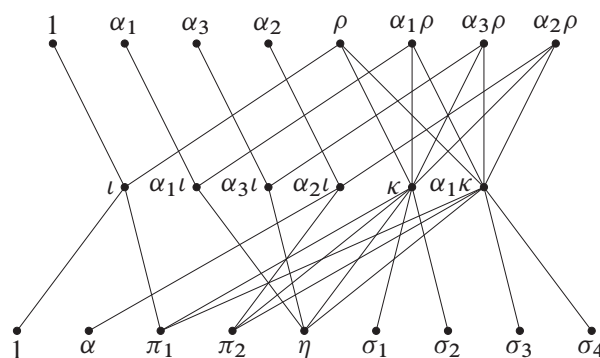
We will call the fusion category tensor generated by ρ in the above theorem \mathcal{AH}_4 , since we will show below that it is \mathcal{AH}_4 .

Remark 4.10. By looking at the details of the proof of Theorem 4.3, one can give a more direct construction of \mathcal{AH}_4 . The way that Theorem 4.3 is proven is by using the structure constants A, ϵ , and η to define an endomorphism ρ and a G -action α of a Cuntz algebra and then completing this Cuntz algebra to a von Neumann algebra with respect to an appropriate state. Then all of the properties in Theorem 4.3 are satisfied except possibly for the action of G being outer. One then needs to make an extra modification in order to enforce outerness. In our case, this extra modification is exactly cancelled out by the de-equivariantization. Since ϵ_z is trivial (which means that $\alpha_{(0,1)}$ acts trivially on the Cuntz algebra), the category that one gets on the von Neumann algebra completion of the Cuntz algebra without any modification is already \mathcal{AH}_4 . Thus it is not strictly necessary to go through the orbifold construction here.

4.5. Reconstruction of the Asaeda–Haagerup subfactor. To show that the fusion category $\mathcal{AH}_{4'}$ constructed in the previous subsection satisfies the defining characterization of \mathcal{AH}_4 (and therefore show that \mathcal{AH}_{4-6} exist), it remains to show that $\mathcal{AH}_{4'}$ is Morita equivalent to \mathcal{AH}_{1-3} . We will show this by deducing that the dual category of $\mathcal{AH}_{4'}$ with respect to a Q-system for $[\text{id}] \oplus [\rho]$ is \mathcal{AH}_2 . In the process, we will arrive at a new construction of the Asaeda–Haagerup subfactor.

Lemma 4.11. *The fusion category $\mathcal{AH}_{2'}$ of endomorphisms of N tensor generated by \bar{u} has a non-trivial invertible object α satisfying $\alpha_2 \iota = \iota \alpha$.*

Lemma 4.12. *The principal graph pair of ι is either*



Proof. The principal graph (which is the upper half of these graphs) is easy to deduce. We discuss the calculation of the dual graphs. Let α be the non-trivial automorphism determined by Lemma 4.11. Hence, $\bar{\iota}\alpha_2\iota = \bar{\iota}\alpha$. For any $g, h \in \mathbb{Z}/4\mathbb{Z}$ we have

$$(\bar{\iota}\alpha_g\iota, \bar{\iota}\alpha_h\iota) = (\alpha_g\bar{\iota}\iota, \bar{\iota}\alpha_h) = (\alpha_g + \alpha_g\rho, \alpha_h + \rho\alpha_h) = \delta_{g,h} + \delta_{g,-h}.$$

This means that ι and $\alpha_2\iota$ are each connected to two vertices with no overlap and $\alpha_1\iota$ and $\alpha_3\iota$ are each connected to the same single vertex. It is clear that 1 is connected to ι and α is connected to $\alpha_2\iota$. We label the other vertex touching ι by π_1 and the other vertex touching $\alpha_2\iota$ by π_2 (in the end the dual will be \mathcal{AH}_2 , so we have named our variables appropriately). Note that, by the defining property of α , we have that $\pi_1\alpha = \pi_2$.

We also compute

$$(\iota\pi_1, \kappa) = (\iota(1 + \pi_1), \kappa) = (\bar{\iota}\iota, \kappa) = 1,$$

so π_1 is connected to κ by a single edge. In a similar way, we find that π_1, π_2 , and η are each connected to each of κ and $\alpha_1\kappa$ by a single edge.

Next we note that we must have $(\rho\kappa, \kappa) + (\rho\kappa, \alpha_1\kappa) = 7$, since $(\rho\kappa, \alpha_g\iota) = 1$ for each g and $d(\rho) = 4d(\iota) + 7d(\kappa)$. Then

$$(\bar{\iota}\kappa, \bar{\iota}\kappa) + (\bar{\iota}\kappa, \bar{\iota}\alpha_1\kappa) = (\bar{\iota}\iota, \kappa\bar{\kappa}) + (\bar{\iota}\iota, \alpha_1\kappa\bar{\kappa}) = (1 + \rho, \kappa\bar{\kappa}) + (1 + \rho, \alpha_1\kappa\bar{\kappa}) = 8.$$

Therefore the total number of paths from κ to itself in the dual graph plus the total number of paths from κ to $\alpha_1\kappa$ must equal 8, and the only two possibilities are pictured. \square

We refer to these two cases as the “four-sigma case” and “one-sigma case”, respectively. We would like to rule out the one sigma case.

Remark 4.13. It is not difficult to deduce the fusion rules for $\mathcal{AH}_{2'}$ in the one-sigma case, getting a fusion ring R . There does exist a consistent AH_4 – R fusion bimodule giving Lemma 4.12. Therefore it is impossible to rule out this possibility from the fusion rules for this bimodule and the two fusion rings alone. As usual, we exploit the additional combinatorial structure coming from looking at multiple subfactors simultaneously. In particular, we look at the compatibility between this bimodule and the bimodule coming from the algebra $1 + \alpha_1\rho$.

Remark 4.14. If one only knows the fusion rules for $\mathcal{AH}_{4'}$ and the fusion module corresponding to ι but not Lemma 4.11, then there are two additional possibilities for the dual graph beyond the above two.

Since $[\text{id}] \oplus [\alpha_1\rho]$ has a Q-system, there is another module over $\mathcal{AH}_{4'}$ with a simple object λ such that $\lambda\bar{\lambda} \cong \text{id} \oplus \alpha_1\rho$. In other words, there is another Type III factor N_1 and a finite-index homomorphism $\lambda : N \rightarrow M$ with dual homomorphism $\bar{\lambda} : M \rightarrow N$ such that $\lambda\bar{\lambda} \cong \text{id} \oplus \alpha_1\rho$.

Lemma 4.15. *The following statements hold.*

- (1) *The $(d + 1)$ -dimensional homomorphisms $\bar{\iota}\alpha_1\iota \in \text{End}_0(N)$ and $\bar{\lambda}\iota \in \text{Hom}_0(N, N_1)$ are irreducible.*
- (2) *The $4d$ -dimensional homomorphism $\bar{\iota}\kappa \in \text{End}_0(N)$ and $\bar{\lambda}\kappa \in \text{Hom}_0(N, N_1)$ satisfy either*
 - (a) *$(\bar{\iota}\kappa, \bar{\iota}\kappa) = 5$ and $(\bar{\lambda}\kappa, \bar{\lambda}\kappa) = 4$, or*
 - (b) *$(\bar{\iota}\kappa, \bar{\iota}\kappa) = 4$ and $(\bar{\lambda}\kappa, \bar{\lambda}\kappa) = 5$.*

Proof. By Frobenius reciprocity,

$$(\bar{\alpha}_1 \iota, \bar{\alpha}_1 \iota) = (\alpha_1 \bar{\iota}, \bar{\iota} \alpha_1) = (\alpha_1 + \alpha_1 \rho, \alpha_1 + \rho \alpha_1) = 1$$

and

$$(\bar{\lambda} \iota, \bar{\lambda} \iota) = (\bar{\iota}, \bar{\lambda} \bar{\lambda}) = (1 + \rho, 1 + \alpha_1 \rho) = 1.$$

Similarly,

$$(\bar{\iota} \kappa, \bar{\iota} \kappa) = (\bar{\iota}, \kappa \bar{\kappa}) = 1 + (\rho \kappa, \kappa)$$

and

$$(\bar{\lambda} \kappa, \bar{\lambda} \kappa) = (\bar{\lambda} \bar{\lambda}, \kappa \bar{\kappa}) = 1 + (\alpha_1 \rho \kappa, \kappa). \quad \square$$

Lemma 4.16. *The one-sigma case of Lemma 4.12 does not occur.*

Proof. Since the objects in $\mathcal{AH}_{2'}$ would have dimensions $(1, 1, d-1, d, d, d+1)$, there is no way to combine them to get an object with dimension $4d$ and with 5-dimensional endomorphism space. Hence, by Lemma 4.15, we must have that $\bar{\iota} \kappa$ satisfies $(\bar{\iota} \kappa, \bar{\iota} \kappa) = 4$ and thus $(\bar{\lambda} \kappa, \bar{\lambda} \kappa) = 5$. We will examine the bimodule category corresponding to $\bar{\lambda} \iota$ more closely.

First note that $d(\bar{\lambda} \iota) = 5 + \sqrt{17}$ which lies in the field $\mathbb{Q}(\sqrt{17})$. Since eigentheory works over any field, it follows that all the other odd vertices must also have dimension lying in $\mathbb{Q}(\sqrt{17})$. We now look more closely at $\bar{\lambda} \kappa$. Frobenius reciprocity shows that $(\bar{\lambda} \kappa, \bar{\iota} \kappa) = 1$ and $(\bar{\lambda} \kappa, \bar{\iota} \alpha \kappa) = 1$, thus $\bar{\lambda} \kappa$ breaks up as $[\bar{\iota} \kappa]$ plus $[\bar{\iota} \alpha \kappa]$ plus three other simple objects. These three objects have total dimension $(3 + \sqrt{17})$ so at least one of them has dimension below $\frac{2}{3}(3 + \sqrt{17})$. Thus, there is a non-trivial division algebra in $\mathcal{AH}_{2'}$ which has dimension smaller than $\frac{4}{9}(3 + \sqrt{17})^2$ and whose dimension is a square in $\mathbb{Q}(\sqrt{17})$. In any division algebra the trivial appears with multiplicity one and invertible objects appear with multiplicity at most one. Thus, from the dimensions of the simple objects in $\mathcal{AH}_{2'}$ the list of possible dimensions of algebra objects in $\mathcal{AH}_{2'}$ below $\frac{4}{9}(3 + \sqrt{17})^2$ is

$$1, 2, d, d+1, d+2, d+3, 2d-1, 2d, 2d+1, 2d+2, 2d+3, 2d+4.$$

Clearly, 2 is not a square in $\mathbb{Q}(\sqrt{17})$, and none of the other numbers are squares either since their norms are the nonsquare numbers $-1, 8, 19, 32, -19, -4, 13, 32, 53$, and 76 , respectively. Thus, the one-sigma case cannot occur. \square

Now that we know that we are in the four-sigma case, we want to work out the fusion rules for the σ_i to show that $\mathcal{AH}_{2'}$ must have a Q -system giving the Asaeda–Haagerup subfactor.

Lemma 4.17. *We have $\alpha \pi_1 = \pi_1 \alpha = \pi_2$.*

Proof. Counting paths from ι to itself through the upper and lower graphs, we see that π_1 and π_2 are both self-dual. Since α is also self-dual, this means that α commutes with π_1 and π_2 . Recall that $1 + \pi_1 = \bar{\iota}$, so $\alpha + \pi_1 \alpha = \bar{\iota} \alpha = \bar{\iota} \alpha_2 \iota = \alpha + \pi_2$. Thus $\pi_1 \alpha = \pi_2$. \square

In keeping with the suggestive notation from AH_2 , we will let $\pi = \pi_2$ and now denote π_1 by $\alpha \pi$.

Lemma 4.18. *The category $\mathcal{AH}_{2'}$ has a self-dual simple object σ with dimension $d(\sigma) = \frac{d-1}{2}$ such that $\sigma^2 = 1 + \sigma + \pi$.*

Proof. Each of the four σ_i has dimension

$$d(\sigma) = \frac{d-1}{2} = \frac{3+\sqrt{17}}{2}.$$

Note that of σ_1 and σ_2 , one object must be self-dual while the other one is not, in order for κ to admit the same number of length-2 paths through the upper graph and through the lower graph. Similarly, one of σ_3 and σ_4 is self-dual and the other is not. Without loss of generality, suppose σ_1 and σ_3 are the self-dual objects while σ_2 and σ_4 are dual to each other. We would like to understand how tensoring with α on the left or right acts on the σ_i . We have

$$\alpha \bar{\iota} \kappa = \bar{\iota} \alpha \kappa = \bar{\alpha}_2 \bar{\iota} \kappa = \bar{\iota} \alpha_2 \kappa = \bar{\iota} \kappa,$$

so tensoring with α on the left fixes $\sigma_1 + \sigma_2$, and similarly fixes $\sigma_3 + \sigma_4$.

From the graph we have

$$\alpha \pi + \pi + \eta + \sigma_1 + \sigma_2 = \bar{\iota} \sigma_1 = (1 + \alpha \pi) \sigma_1 = \sigma_1 + \alpha \pi \sigma_1,$$

and hence $\alpha \pi \sigma_1 = \alpha \pi + \pi + \eta + \sigma_2$. By Frobenius reciprocity, it follows that $(\pi \sigma_1, \alpha \sigma_2) = 1$ and $(\pi, \alpha \sigma_2 \sigma_1) = 1$.

We claim that $\alpha \sigma_2 = \sigma_1$. Suppose to the contrary that $\alpha \sigma_2 = \sigma_2$. Then $\sigma_2 \sigma_1$ contains π . Thus the remaining part of $\sigma_2 \sigma_1$ has dimension

$$d(\sigma)^2 - d(\pi) = \frac{5 + \sqrt{17}}{2},$$

which means it must be the sum of an invertible object and one of the σ_i . But this is impossible because $\sigma_2 \sigma_1$ does not contain 1 since they are not dual to each other, and does not contain α because $\alpha \sigma_2 = \sigma_2$ is not dual to σ_1 . Therefore $\alpha \sigma_2 = \sigma_1$, and similarly $\alpha \sigma_3 = \sigma_4$.

We set $\sigma = \sigma_1$, so that $\alpha \sigma = \sigma_2$, $\sigma \alpha = \bar{\sigma}_2 = \sigma_4$, and $\alpha \sigma \alpha = \alpha \sigma_4 = \sigma_3$. Now return to σ^2 , we now know that $1 = (\pi \sigma_1, \alpha \sigma_2) = (\pi \sigma, \sigma)$ so we must have that π occurs in σ^2 . Thus, by dimension considerations, we must have that $\sigma^2 = 1 + \sigma_i + \pi$ and since σ is selfdual while $\alpha \sigma$ and $\sigma \alpha$ are not, we must have $\sigma^2 = 1 + \sigma + \pi$ or $\sigma^2 = 1 + \alpha \sigma \alpha + \pi$. So our final task is the rule out the latter possibility.

Suppose that $x = \alpha \sigma \alpha$. Then from the relation

$$\sigma^2 \sigma = \sigma \sigma^2$$

we get

$$\sigma \pi + \sigma \alpha \sigma \alpha = \pi \sigma + \alpha \sigma \alpha \sigma.$$

Since $\pi \sigma = \pi + \alpha \pi + \eta + \sigma$ and π and σ are both self-dual, we have $\sigma \pi = \bar{\sigma} \bar{\pi} = \bar{\pi} \bar{\sigma} = \pi \sigma$. Therefore, $\sigma \alpha \sigma \alpha = \alpha \sigma \alpha \sigma$. So consider $\sigma \alpha \sigma$, since its dimension is half odd (that is it is not in $\mathbb{Z}[\sqrt{17}]$), it must have an odd number of σ_i in it. But conjugating by α acts faithfully on the σ_i , so $\sigma \alpha \sigma \alpha$ and $\alpha \sigma \alpha \sigma$ cannot have the same σ_i summands. This is a contradiction, so we must have $x = \sigma$ and $\sigma^2 = 1 + \pi + \sigma$. \square

This gives an alternative proof for the existence of the Asaeda–Haagerup subfactor [2] by applying a short skein theoretic argument from [10].

Theorem 4.19. *The object $1 \oplus \sigma$ in \mathcal{AH}_2 has an algebra structure giving the Asaeda–Haagerup subfactor.*

Proof. Since the fusion ring contains a self-dual simple object σ with

$$d(\sigma) = \frac{d-1}{2} = \frac{3+\sqrt{17}}{2}$$

satisfying $\sigma^2 = 1 + \sigma + \pi$ with π simple, $1 + \sigma$ admits a Q-system by [10, Section 3.2]. It is easy to see that the only possible graph pair for such a Q-system is given by the Asaeda–Haagerup principal graphs. \square

Corollary 4.20. *We have that $\mathcal{AH}_{2'}$ is equivalent to \mathcal{AH}_2 .*

Proof. This follows from the uniqueness of the connection on the Asaeda–Haagerup principal graph pair. \square

Theorem 4.21. *The fusion categories \mathcal{AH}_{4-6} exist. The Morita equivalence class contains exactly six fusion categories up to equivalence.*

Proof. Since $\mathcal{AH}_{4'}$ is Morita equivalent to \mathcal{AH}_2 via a fusion category with fusion module is $L_{AH_2}^{(4)}$, it satisfies the characterizing property of \mathcal{AH}_4 . Thus \mathcal{AH}_4 exists, and so by Theorem 3.17 each of \mathcal{AH}_{4-6} exist and are unique. By Theorem 3.24, there are no fusion categories realizing case (7), so by Theorem 2.19 the only fusion categories in the Asaeda–Haagerup Morita equivalence class are \mathcal{AH}_i for $1 \leq i \leq 6$. \square

5. Applications

In this section we outline several applications of our main theorems. These applications will appear in subsequent papers.

5.1. Drinfeld center. The Drinfeld center, or quantum double, $Z(\mathcal{C})$ of a spherical fusion category \mathcal{C} is the braided fusion category whose objects are pairs of an object in $X \in \mathcal{C}$ endowed with a natural half-braiding $\eta_X : X \otimes Y \rightarrow Y \otimes X$. In the subfactor setting, the Drinfeld center can be realized as one of the even parts of the asymptotic inclusion or the Longo–Rehren construction. The category $Z(\mathcal{C})$ is a modular tensor category, in the sense that the S -matrix given by the (normalized) traces of the Hopf links coming from the braiding is invertible. The S -matrix, along with the diagonal T -matrix which gives the twist of each object in the modular tensor category, is called the modular data.

Since the discovery of the “exotic” Haagerup and Asaeda–Haagerup in the 1990s, there has been significant interest in computing their quantum doubles. Izumi developed a general method to describe the quantum double of a fusion category from Cuntz algebra models in terms of tube algebras [12]; he used this to describe the quantum doubles of the Haagerup subfactor and its generalizations [13]. The modular data of the Haagerup subfactor was simplified and generalized by Evans and Gannon [5]. By contrast, for the Asaeda–Haagerup subfactor, the quantum double has remained elusive.

Theorem 4.9 gives a Cuntz algebra model for the fusion category \mathcal{AH}_4 , and we can apply the method of [12] to describe its tube algebra. In particular, we can compute the simple objects, find their dimensions, and compute the modular data of the Asaeda–Haagerup subfactor. This

calculation is complicated, so we will defer the details to the subsequent paper [7] and provide a summary of the results here.

The rank of the quantum double of the Asaeda–Haagerup subfactor is 22. In addition to the trivial object, there are six objects of dimension $1 + 4(4 + \sqrt{17})$, eight objects of dimension $8(4 + \sqrt{17})$, one of dimension $1 + 8(4 + \sqrt{17})$, and six of dimension $2 + 8(4 + \sqrt{17})$. Following Evans–Gannon, as in the Haagerup case one can choose a nice ordering of the objects so that the modular data breaks up cleanly into blocks. In this ordering, the first fourteen entries of the diagonal of the T -matrix are given by the vector $(1, 1, 1, -1, 1, -1, i, -i, 1, 1, 1, 1, 1, 1)$; the final eight entries are given by $e^{6l^2\pi i/17}$, $1 \leq l \leq 8$. Similarly, the S -matrix is given by the following blocks. There is a 14×14 block given by

$$\frac{1}{8} \begin{pmatrix} \frac{8}{\Lambda} & \frac{8d^2}{\Lambda} & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{8d^2}{\Lambda} & \frac{8}{\Lambda} & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & -4 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & -2 & -2 \\ 2 & 2 & -4 & 4 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & -2 & -2 \\ 2 & 2 & 0 & 0 & 4 & -4 & 0 & 0 & -2 & -2 & 2 & 2 & -2 & -2 \\ 2 & 2 & 0 & 0 & -4 & 4 & 0 & 0 & -2 & -2 & 2 & 2 & -2 & -2 \\ 2 & 2 & 0 & 0 & 0 & 0 & -4 & 4 & -2 & -2 & -2 & -2 & 2 & 2 \\ 2 & 2 & 0 & 0 & 0 & 0 & 4 & -4 & -2 & -2 & -2 & -2 & 2 & 2 \\ 1 & 1 & 2 & 2 & -2 & -2 & -2 & -2 & 5 & -3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & -2 & -2 & -2 & -2 & -3 & 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & -2 & 2 & 2 & -2 & -2 & 1 & 1 & 5 & -3 & 1 & 1 \\ 1 & 1 & -2 & -2 & 2 & 2 & -2 & -2 & 1 & 1 & -3 & 5 & 1 & 1 \\ 1 & 1 & -2 & -2 & -2 & -2 & 2 & 2 & 1 & 1 & 1 & 1 & 5 & -3 \\ 1 & 1 & -2 & -2 & -2 & -2 & 2 & 2 & 1 & 1 & 1 & 1 & -3 & 5 \end{pmatrix},$$

where $d = 4 + \sqrt{17}$ and $\Lambda = 4(1 + d^2)$ is the global dimension. Then there is an 8×8 block given by

$$S_{14+k, 14+l} = -\frac{2}{\sqrt{17}} \cos\left(\frac{12\pi lk}{17}\right), \quad 1 \leq k, l \leq 8.$$

These blocks only interact in the first two rows/columns. For $15 \leq j \leq 22$, we have

$$S_{1j} = \frac{1}{\sqrt{17}}, \quad S_{2j} = -\frac{1}{\sqrt{17}}, \quad S_{ij} = 0, \quad 1 \leq i \leq 14.$$

This modular data has a very similar form to that of the Haagerup subfactor, and there is hope that it can be generalized into a series as Evans and Gannon did for the Haagerup modular data.

We note that Morrison and Walker have found a purely combinatorial method to find the dimensions of the simple objects in the quantum double of the Asaeda–Haagerup subfactor as well as the induction functors to \mathcal{AH}_1 and \mathcal{AH}_2 (see [18]). However, their method does not provide an explicit description of the quantum double or give the modular data.

5.2. Graded extensions. A grading on a fusion category \mathcal{D} by a group G is a decomposition $\mathcal{D} \cong \bigoplus_{g \in G} \mathcal{D}_g$ such that the tensor product of an object in \mathcal{D}_g and an object in \mathcal{D}_h

lands in \mathcal{D}_{gh} . Such a grading is called faithful if the \mathcal{D}_g are all nonzero. A graded extension of a fusion category \mathcal{C} by a group G is a faithfully G -graded fusion category \mathcal{D} such that the identity component \mathcal{D}_0 is \mathcal{C} .

Graded extensions by the group $\mathbb{Z}/2\mathbb{Z}$ are important in subfactor theory, where it is natural to ask whether there is a $\mathbb{Z}/2\mathbb{Z}$ extension of the even part of the subfactor such that \mathcal{D}_1 is the odd part of the subfactor. That is, we want to know whether one can find an isomorphism of factors M and N which identifies the two even parts together and the two odd parts together. In planar algebraic language, this is asking about whether there is an unshaded version of the planar algebra attached to a subfactor. For example, for group subfactors, such a $\mathbb{Z}/2\mathbb{Z}$ extension is exactly a Tambara–Yamagami category.

Graded extensions of a fusion category \mathcal{C} by a finite group G were classified in [4] using homotopy theory. The Brauer–Picard 3-group (also called a categorical 2-group) of \mathcal{C} is the full subgroupoid of the Brauer–Picard 3-groupoid whose only object is \mathcal{C} . (In other words, its only object is \mathcal{C} , its 1-morphisms are Morita auto-equivalences, and its 2- and 3-morphisms are defined as in the Brauer–Picard groupoid.) Then, giving a G extension of \mathcal{C} is the same thing as giving a map of 3-groups from G to the Brauer–Picard 3-group of \mathcal{C} . Furthermore, using obstruction theory from algebraic topology, such maps can be classified homologically. By the usual change-of-basepoint conjugation, a Morita equivalence between \mathcal{C} and \mathcal{D} induces an equivalence between their Brauer–Picard 3-groups. Therefore, to understand the graded extensions of \mathcal{C} , it suffices to understand the graded extensions of any fusion category Morita equivalent to \mathcal{C} .

In [8], we have given a partial description of the Brauer–Picard 3-group of the Asaeda–Haagerup fusion categories. In particular, its homotopy groups are $\pi_1 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\pi_2 = 0$, and $\pi_3 = \mathbb{C}^\times$. For Brauer–Picard 3-groups of fusion categories, π_1 automatically acts trivially on π_3 . Thus in order to completely describe the Brauer–Picard 3-group of Asaeda–Haagerup, we need only understand a single Postnikov k -invariant living in $H^4(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$. Furthermore, given a map $G \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, whether a G extension compatible with that map exists depends only on whether this k -invariant becomes trivial in $H^4(G, \mathbb{C}^\times)$. In particular, since $H^4(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$ is trivial, the obstruction automatically vanishes for $\mathbb{Z}/2\mathbb{Z}$ extensions. Therefore, any Morita autoequivalence of one of the Asaeda–Haagerup fusion categories gives rise to a $\mathbb{Z}/2\mathbb{Z}$ -graded extension. (There are two such extensions, since they form a torsor for $H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$.) A natural question is whether there exist extensions by the full Brauer–Picard group, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In [8], we were unable to resolve this question, since unlike in the case of cyclic groups, $H^4(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$ is non-trivial. In particular, we need to know the actual value of the k -invariant in this non-trivial homology group.

The results of this paper will allow us to answer these question. Instead of trying to understand $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extensions of the original Asaeda–Haagerup fusion categories, we can instead consider extensions of the new category \mathcal{AH}_4 which has a computationally accessible model as endomorphisms of the Cuntz algebra \mathcal{O}_9 . Furthermore, for \mathcal{AH}_4 , every Morita autoequivalence is realized by an outer automorphism by Theorem 3.18, so the graded components of any extension are all trivial as \mathcal{AH}_4 -module categories. To realize such an extension, we only need to find a collection of endomorphisms of a factor containing a copy of \mathcal{AH}_4 and four additional automorphisms which implement the outer automorphisms of \mathcal{AH}_4 .

In a subsequent paper we will explicitly construct the outer automorphisms of \mathcal{AH}_4 , and construct a (non-trivial) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -graded extension of \mathcal{AH}_4 . By the extension theory of [4], this implies the existence of non-trivial $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -graded extensions for each \mathcal{AH}_i

for $1 \leq i \leq 6$. In particular, we can conclude that the k -invariant vanishes, so the Brauer–Picard 3-group of the Asaeda–Haagerup subfactor splits as a product of Eilenberg–MacLane spaces $K(\mathbb{Z}/2 \times \mathbb{Z}/2, 1) \times K(\mathbb{C}^\times, 3)$. As a consequence, extensions by any group G are classified by pairs of a homomorphism from G to the Klein 4-group together with an element of $H^3(G, \mathbb{C}^\times)$.

5.3. Intermediate subfactors. From a complete understanding of the Brauer–Picard groupoid of \mathcal{C} , one can read off all division algebras in \mathcal{C} as internal endomorphisms of objects in the module categories over \mathcal{C} . In subfactor language, this is the list of all irreducible overfactors of N which can be built from a fixed finite collection of N – N bimodules. Each division algebra in \mathcal{C} corresponds to the internal endomorphisms for some simple object η in a module category in \mathcal{C} , and two objects give the same algebra if they are in the same orbit under the action of invertible objects in the dual category. Thus one can first write down all the module categories (each module category gives a Morita equivalence to its dual category, and this is well-defined up to outer automorphism of the dual category) and then find the orbits of simple objects. For the Asaeda–Haagerup subfactor our results immediately imply that we have the following number of division algebras A in each of the \mathcal{AH}_i such that the dual category of A – A bimodules is \mathcal{AH}_j . (For $1 \leq i, j \leq 3$ all of these subfactors were described in the appendix to [10].)

	\mathcal{AH}_1	\mathcal{AH}_2	\mathcal{AH}_3	\mathcal{AH}_4	\mathcal{AH}_5	\mathcal{AH}_6
\mathcal{AH}_1	20	12	12	2	2	2
\mathcal{AH}_2	18	14	14	2	2	2
\mathcal{AH}_3	18	14	14	2	2	2
\mathcal{AH}_4	20	12	12	2	2	2
\mathcal{AH}_5	16	16	16	2	2	2
\mathcal{AH}_6	20	12	12	2	2	2

Table 1. Number of division algebras in \mathcal{AH}_i with dual \mathcal{AH}_j .

However, one can go further and use the composition rules in the Brauer–Picard groupoid to understand all inclusions between all of these algebras, and thus all lattices of intermediate subfactors. We did so in [9] for the Haagerup fusion categories. Here is a quick sketch of where this relationship comes from. Suppose that $1 \subset A \subset B$ is an inclusion of division algebras. Let \mathcal{D} be the fusion category of A – A bimodules and \mathcal{E} the fusion category of B – B bimodules. Let \mathcal{L} be the \mathcal{C} – \mathcal{D} bimodule category of 1 – A bimodules, let \mathcal{M} be the \mathcal{D} – \mathcal{E} bimodule category of A – B bimodules, and let \mathcal{N} be the \mathcal{C} – \mathcal{E} bimodule category of 1 – B bimodules. Then $B = A \otimes_A B$ shows that under the tensor product map $\mathcal{L} \boxtimes_{\mathcal{D}} \mathcal{M} \rightarrow \mathcal{N}$ the tensor product of two simple objects is simple. Thus intermediate subfactors yield a factorization of an object in some bimodule category as a tensor product of two other objects in bimodule categories. In the other direction, if $\iota = \eta\psi$ is a decomposition of a simple object ι in a bimodule category as a tensor product of objects in (possibly different) bimodule categories, then we get an inclusion of division rings $1 \subset \eta\bar{\eta} = \eta 1 \bar{\eta} \subset \eta\psi\bar{\psi}\bar{\eta} = \iota\bar{\iota}$. Thus there is a correspondence between inclusions of algebras and factorizations of objects in bimodule categories. In full generality,

this correspondence can get a bit delicate as automorphisms of \mathcal{D} (both inner and outer!) play a role, as does the group π_2 which measures the ambiguity of the tensor product.

In a later paper, we hope to give a full description of this recipe for reading off lattices of intermediate subfactors from tensor product rules between bimodule categories, and to apply this to give a classification of all intermediate subfactor lattices for all division algebras in the Asaeda–Haagerup fusion categories. It is already known that there are several interesting quadrilaterals involve AH , $AH + 1$ and $AH + 2$, and we hope to find more. Such a calculation will require computing some additional compositions which we have not yet worked out.

A. Fusion rules

In this appendix we list the fusion modules corresponding to the four cases in Theorem 2.19. Each case corresponds to a triple of fusion modules $(K_{AH_1}, L_{AH_2}, M_{AH_3})$. We first recall the fusion rules of AH_{1-3} , given by the following tables from [10].

The Asaeda–Haagerup fusion category \mathcal{AH}_1 has six simple objects, which we will order $1, \chi, \psi, \tau, \zeta, \sigma$, and have dimensions $1, \frac{d+1}{2}, \frac{d-1}{2}, \frac{3d-1}{2}, d, \frac{d+3}{2}$, respectively, where $d = 4 + \sqrt{17}$. We use the abbreviation $\Lambda = \psi + \chi + \sigma + \zeta + \tau$. Since the multiplication is commutative, we omit the sub-diagonal entries.

	ψ	χ	σ	ζ	τ
ψ	$1 + \psi + \zeta$	$\chi + \tau$	$\zeta + \tau$	$\psi + \sigma + \zeta + \tau$	$\chi + \sigma + \zeta + 2\tau$
χ		$1 + \psi + \chi + \tau$	$\sigma + \zeta + \tau$	$\sigma + \zeta + 2\tau$	$\Lambda + \zeta + \tau$
σ			$1 + \chi + \sigma + \zeta + \tau$	$\Lambda + \tau$	$\Lambda + \zeta + 2\tau$
ζ				$1 + \Lambda + \zeta + 2\tau$	$\Lambda + \chi + \sigma + 2\zeta + 3\tau$
τ					$1 + 2\Lambda + \sigma + 2\zeta + 4\tau$

Table 2. AH_1 multiplication table.

The Asaeda–Haagerup fusion category \mathcal{AH}_2 has nine simple objects, which we will order $1, \alpha, \rho, \alpha\rho, \rho\alpha, \alpha\rho\alpha, \pi, \alpha\pi, \eta$, and have dimensions $1, 1, \frac{d-1}{2}, \frac{d-1}{2}, \frac{d-1}{2}, \frac{d-1}{2}, d, d, d + 1$, respectively. The rules involving α are

$$\alpha^2 = 1, \quad \pi\alpha = \alpha\pi, \quad \alpha\eta = \eta\alpha = \eta, \quad \rho\alpha\rho = \alpha\rho\alpha + \eta.$$

We use the abbreviations $\Gamma = \pi + \alpha\pi + \eta$, $\Delta = \rho + \alpha\rho + \rho\alpha + \alpha\rho\alpha$.

	ρ	π	η
ρ	$1 + \rho + \pi$	$\rho + \Gamma$	$\alpha\rho + \alpha\rho\alpha + \Gamma$
π	$\rho + \Gamma$	$1 + \Delta + 2\Gamma$	$\Delta + 2\Gamma + \eta$
η	$\rho\alpha + \alpha\rho\alpha + \Gamma$	$\Delta + 2\Gamma + \eta$	$1 + \alpha + \Delta + 2\Gamma + \pi + \alpha\pi$

Table 3. AH_2 partial multiplication table.

The Asaeda–Haagerup fusion category \mathcal{AH}_3 has nine simple objects, which we will order $1, \beta, \xi, \beta\xi, \xi\beta, \beta\xi\beta, \mu, \beta\mu, \nu$, and have dimensions $1, 1, \frac{d+1}{2}, \frac{d+1}{2}, \frac{d+1}{2}, \frac{d+1}{2}, d, d, d-1$, respectively. The rules involving β are

$$\beta^2 = 1, \quad \mu\beta = \beta\mu, \quad \beta\nu = \nu\beta = \nu, \quad \xi\beta\xi = \beta\xi\beta + \mu + \beta\mu.$$

We use the abbreviations $\Pi = \mu + \beta\mu + \nu$, $\Psi = \xi + \beta\xi + \xi\beta + \beta\xi\beta$.

	ξ	μ	ν
ξ	$1 + \xi + \mu + \nu$	$\xi + \beta\xi + \beta\xi\beta + \Pi$	$\xi + \xi\beta + \Pi$
μ	$\xi + \xi\beta + \beta\xi\beta + \Pi$	$1 + \Pi + 2\Psi$	$\Pi + \Psi + \mu + \beta\mu$
ν	$\xi + \beta\xi + \Pi$	$\Pi + \Psi + \mu + \beta\mu$	$1 + \beta + \Pi + \Psi + \nu$

Table 4. \mathcal{AH}_3 partial multiplication table.

Each fusion module will be represented as n adjacent $m \times n$ integer matrices (separated by vertical lines), where m is the number of basis elements of \mathcal{AH}_l and n is the number of basis elements of the module category K . The ij th entry of the k th matrix gives the multiplicity $(R_k X_i, R_j)$, where the $\{R_j\}$ are basis elements of K and the X_i are basis elements of \mathcal{AH}_l . For each case the fusion modules are listed in the following order: first $K_{\mathcal{AH}_1}^{(i)}$, then $L_{\mathcal{AH}_2}^{(i)}$, then $M_{\mathcal{AH}_3}^{(i)}$. With this notation, here are the four cases. Recall that cases (4)–(7) correspond to cases (c), (a), (b), and (d) respectively in [10, Theorem 6.10].

Case (4). We have

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c} L_{\mathcal{AH}_1}^{(4)} & \theta_{41}^1 & \theta_{41}^2 & \theta_{41}^3 & \theta_{41}^4 & \theta_{41}^5 & \theta_{41}^1 & \theta_{41}^2 & \theta_{41}^3 & \theta_{41}^4 & \theta_{41}^5 & \theta_{41}^1 & \theta_{41}^2 & \theta_{41}^3 & \theta_{41}^4 & \theta_{41}^5 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \chi & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ \psi & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \tau & 2 & 2 & 2 & 1 & 3 & 2 & 2 & 1 & 2 & 3 & 2 & 1 & 2 & 2 & 2 & 3 \\ \zeta & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 \\ \sigma & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline L_{\mathcal{AH}_1}^{(4)} & \theta_{41}^1 & \theta_{41}^2 & \theta_{41}^3 & \theta_{41}^4 & \theta_{41}^5 & \theta_{41}^1 & \theta_{41}^2 & \theta_{41}^3 & \theta_{41}^4 & \theta_{41}^5 & & & & & & \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & & \\ \chi & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & & & & & & \\ \psi & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & \\ \tau & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & & & & & & \\ \zeta & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & & & & & & \\ \sigma & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & & & & & & \end{array} \right),$$

$$\begin{pmatrix}
M_{\mathcal{AH}_2}^{(4)} & \begin{array}{c|cccc} \theta_{42}^1 \otimes - & & & \end{array} & \begin{array}{c|cccc} \theta_{42}^2 \otimes - & & & \end{array} & \begin{array}{c|cccc} \theta_{42}^3 \otimes - & & & \end{array} \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\rho & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\alpha\rho & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\rho\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\alpha\rho\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\pi & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\alpha\pi & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\eta & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
M_{\mathcal{AH}_2}^{(4)} & \begin{array}{c|cccc} \theta_{42}^4 \otimes - & & & \end{array} & \begin{array}{c|cccc} \theta_{42}^5 \otimes - & & & \end{array} & \begin{array}{c|cccc} \theta_{42}^6 \otimes - & & & \end{array} \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\rho & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 2 \\
\alpha\rho & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 \\
\rho\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\
\alpha\rho\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 2 \\
\pi & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 4 & 3 & 1 & 1 & 1 & 4 \\
\alpha\pi & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 4 & 1 & 1 & 1 & 4 \\
\eta & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 1 & 1 & 1 & 4
\end{pmatrix},$$

$$\begin{pmatrix}
N_{\mathcal{AH}_3}^{(4)} & \begin{array}{c|ccc} \theta_{43}^1 \otimes - & & & \end{array} & \begin{array}{c|ccc} \theta_{43}^2 \otimes - & & & \end{array} & \begin{array}{c|ccc} \theta_{43}^3 \otimes - & & & \end{array} \\
\hline
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\beta & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\xi & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 4 \\
\beta\xi & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 4 \\
\xi\beta & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 4 \\
\beta\xi\beta & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 4 \\
\mu & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 2 & 2 & 7 \\
\beta\mu & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 2 & 2 & 7 \\
\nu & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 6
\end{pmatrix}.$$

Case (5). We have

$$\begin{pmatrix}
L_{\mathcal{AH}_1}^{(5)} & \begin{array}{c|ccc} \theta_{51}^1 \otimes - & & & \end{array} & \begin{array}{c|ccc} \theta_{51}^2 \otimes - & & & \end{array} & \begin{array}{c|ccc} \theta_{51}^3 \otimes - & & & \end{array} & \begin{array}{c|ccc} \theta_{51}^4 \otimes - & & & \end{array} \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\chi & 2 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 \\
\psi & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\
\tau & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 3 & 3 & 4 \\
\zeta & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 3 & 2 & 2 & 2 & 3 \\
\sigma & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2
\end{pmatrix},$$

$M_{\mathcal{AH}_2}^{(5)}$	$\theta_{52}^1 \otimes -$						$\theta_{52}^2 \otimes -$						$\theta_{52}^3 \otimes -$					
	θ_{52}^1	θ_{52}^2	θ_{52}^3	θ_{52}^4	θ_{52}^5	θ_{52}^6	θ_{52}^1	θ_{52}^2	θ_{52}^3	θ_{52}^4	θ_{52}^5	θ_{52}^6	θ_{52}^1	θ_{52}^2	θ_{52}^3	θ_{52}^4	θ_{52}^5	θ_{52}^6
1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
α	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
ρ	0	0	0	0	1	1	0	0	1	1	0	0	0	1	1	1	0	1
$\alpha\rho$	0	0	0	0	1	1	0	0	1	1	0	0	1	0	1	0	1	1
$\rho\alpha$	0	0	1	1	0	0	0	0	0	0	1	1	0	1	1	0	1	1
$\alpha\rho\alpha$	0	0	1	1	0	0	0	0	0	0	1	1	1	0	1	1	0	1
π	1	0	1	1	1	1	0	1	1	1	1	1	1	1	2	2	2	1
$\alpha\pi$	1	0	1	1	1	1	0	1	1	1	1	1	1	1	1	2	2	2
η	0	2	1	1	1	1	2	0	1	1	1	1	1	1	2	2	2	2
$M_{\mathcal{AH}_2}^{(5)}$	$\theta_{52}^4 \otimes -$						$\theta_{52}^5 \otimes -$						$\theta_{52}^6 \otimes -$					
	θ_{52}^1	θ_{52}^2	θ_{52}^3	θ_{52}^4	θ_{52}^5	θ_{52}^6	θ_{52}^1	θ_{52}^2	θ_{52}^3	θ_{52}^4	θ_{52}^5	θ_{52}^6	θ_{52}^1	θ_{52}^2	θ_{52}^3	θ_{52}^4	θ_{52}^5	θ_{52}^6
1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1
α	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
ρ	0	1	1	1	1	0	1	0	0	1	1	1	1	0	1	0	1	1
$\alpha\rho$	1	0	0	1	1	1	0	1	1	1	1	0	0	1	1	1	0	1
$\rho\alpha$	0	1	0	1	1	1	1	0	1	1	1	0	1	0	1	1	0	1
$\alpha\rho\alpha$	1	0	1	1	1	0	0	1	0	1	1	1	0	1	1	0	1	1
π	1	1	2	2	1	2	1	1	2	1	2	2	1	1	1	2	2	2
$\alpha\pi$	1	1	2	1	2	2	1	1	2	2	1	2	1	1	2	2	2	1
η	1	1	2	2	2	2	1	1	2	2	2	2	1	1	2	2	2	2
$N_{\mathcal{AH}_3}^{(5)}$	$\theta_{53}^1 \otimes -$						$\theta_{53}^2 \otimes -$						$\theta_{53}^3 \otimes -$					
	θ_{53}^1	θ_{53}^2	θ_{53}^3	θ_{53}^4	θ_{53}^5	θ_{53}^6	θ_{53}^1	θ_{53}^2	θ_{53}^3	θ_{53}^4	θ_{53}^5	θ_{53}^6	θ_{53}^1	θ_{53}^2	θ_{53}^3	θ_{53}^4	θ_{53}^5	θ_{53}^6
1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
β	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
ξ	0	0	1	0	0	1	0	0	0	1	1	0	1	0	0	0	0	1
$\beta\xi$	0	1	0	0	1	0	1	0	0	0	0	1	0	0	0	1	1	0
$\xi\beta$	0	1	0	0	0	1	1	0	0	0	1	0	0	0	0	1	0	1
$\beta\xi\beta$	0	0	1	0	1	0	0	0	0	1	0	1	1	0	0	0	1	0
μ	1	0	0	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1
$\beta\mu$	0	0	0	1	1	1	0	0	1	0	1	1	0	1	0	0	1	1
ν	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1
$N_{\mathcal{AH}_3}^{(5)}$	$\theta_{53}^4 \otimes -$						$\theta_{53}^5 \otimes -$						$\theta_{53}^6 \otimes -$					
	θ_{53}^1	θ_{53}^2	θ_{53}^3	θ_{53}^4	θ_{53}^5	θ_{53}^6	θ_{53}^1	θ_{53}^2	θ_{53}^3	θ_{53}^4	θ_{53}^5	θ_{53}^6	θ_{53}^1	θ_{53}^2	θ_{53}^3	θ_{53}^4	θ_{53}^5	θ_{53}^6
1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1
β	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
ξ	0	1	0	0	1	0	0	1	0	1	2	2	1	0	1	0	2	2
$\beta\xi$	0	0	1	0	0	1	0	1	0	1	2	2	1	0	1	0	2	2
$\xi\beta$	0	0	1	0	1	0	1	0	1	0	2	2	0	1	0	1	2	2
$\beta\xi\beta$	0	1	0	0	0	1	1	0	1	0	2	2	0	1	0	1	2	2
μ	0	0	0	1	1	1	1	1	1	1	3	4	1	1	1	1	4	3
$\beta\mu$	1	0	0	0	1	1	1	1	1	1	3	4	1	1	1	1	4	3
ν	0	0	0	0	1	1	1	1	1	1	4	2	1	1	1	1	2	4

Case (6). We have

$$\left(\begin{array}{c|c|c|c|c|c} L_{\mathcal{AH}_1}^{(6)} & \theta_{61}^1 \otimes - & \theta_{61}^2 \otimes - & \theta_{61}^3 \otimes - & & \\ \hline & \theta_{61}^1 & \theta_{61}^2 & \theta_{61}^3 & \theta_{61}^4 & \theta_{61}^5 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ \chi & 1 & 0 & 0 & 1 & 1 \\ \psi & 0 & 0 & 0 & 1 & 1 \\ \tau & 1 & 1 & 1 & 1 & 3 \\ \zeta & 1 & 1 & 1 & 0 & 2 \\ \sigma & 0 & 1 & 1 & 1 & 1 \\ \hline L_{\mathcal{AH}_1}^{(6)} & \theta_{61}^4 \otimes - & \theta_{61}^5 \otimes - & & & \\ \hline & \theta_{61}^1 & \theta_{61}^2 & \theta_{61}^3 & \theta_{61}^4 & \theta_{61}^5 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ \chi & 1 & 0 & 0 & 1 & 1 \\ \psi & 1 & 0 & 0 & 0 & 1 \\ \tau & 1 & 1 & 1 & 1 & 3 \\ \zeta & 0 & 1 & 1 & 1 & 2 \\ \sigma & 1 & 1 & 1 & 0 & 1 \\ \hline & \theta_{61}^1 & \theta_{61}^2 & \theta_{61}^3 & \theta_{61}^4 & \theta_{61}^5 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 \\ \chi & 1 & 1 & 1 & 1 & 3 \\ \psi & 1 & 1 & 1 & 1 & 2 \\ \tau & 3 & 3 & 3 & 3 & 7 \\ \zeta & 2 & 2 & 2 & 2 & 5 \\ \sigma & 1 & 1 & 1 & 1 & 4 \end{array} \right),$$

$$\left(\begin{array}{c|c|c|c|c|c} N_{\mathcal{AH}_3}^{(6)} & \theta_{63}^1 \otimes - & \theta_{63}^2 \otimes - & \theta_{63}^3 \otimes - & & \\ \hline & \theta_{63}^1 & \theta_{63}^2 & \theta_{63}^3 & \theta_{63}^4 & \theta_{63}^5 & \theta_{63}^6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & 1 & 0 & 0 & 0 & 0 \\ \xi & 1 & 0 & 0 & 0 & 1 & 1 \\ \beta\xi & 0 & 1 & 1 & 1 & 0 & 0 \\ \xi\beta & 0 & 1 & 0 & 0 & 1 & 1 \\ \beta\xi\beta & 1 & 0 & 1 & 1 & 0 & 0 \\ \mu & 1 & 0 & 1 & 1 & 1 & 1 \\ \beta\mu & 0 & 1 & 1 & 1 & 1 & 1 \\ \nu & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline N_{\mathcal{AH}_3}^{(6)} & \theta_{63}^4 \otimes - & \theta_{63}^5 \otimes - & \theta_{63}^6 \otimes - & & \\ \hline & \theta_{63}^1 & \theta_{63}^2 & \theta_{63}^3 & \theta_{63}^4 & \theta_{63}^5 & \theta_{63}^6 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \beta & 0 & 0 & 1 & 0 & 0 & 0 \\ \xi & 0 & 1 & 1 & 1 & 1 & 1 \\ \beta\xi & 0 & 1 & 1 & 1 & 1 & 1 \\ \xi\beta & 1 & 0 & 1 & 1 & 1 & 1 \\ \beta\xi\beta & 1 & 0 & 1 & 1 & 1 & 1 \\ \mu & 1 & 1 & 2 & 1 & 2 & 2 \\ \beta\mu & 1 & 1 & 1 & 2 & 2 & 2 \\ \nu & 1 & 1 & 2 & 2 & 1 & 1 \\ \hline & \theta_{63}^1 & \theta_{63}^2 & \theta_{63}^3 & \theta_{63}^4 & \theta_{63}^5 & \theta_{63}^6 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & 0 & 0 & 0 & 1 \\ \xi & 1 & 0 & 1 & 1 & 1 & 1 \\ \beta\xi & 1 & 0 & 1 & 1 & 1 & 1 \\ \xi\beta & 0 & 1 & 1 & 1 & 1 & 1 \\ \beta\xi\beta & 0 & 1 & 1 & 1 & 1 & 1 \\ \mu & 1 & 1 & 2 & 2 & 1 & 2 \\ \beta\mu & 1 & 1 & 2 & 2 & 2 & 1 \\ \nu & 1 & 1 & 1 & 1 & 2 & 2 \end{array} \right),$$

$$\left(\begin{array}{c|ccc|ccc|ccc} M_{\mathcal{AH}_2}^{(6)} & \theta_{62}^1 \otimes - & & & \theta_{62}^2 \otimes - & & & \theta_{62}^3 \otimes - & & \\ & \theta_{62}^1 & \theta_{62}^2 & \theta_{62}^3 & \theta_{62}^1 & \theta_{62}^2 & \theta_{62}^3 & \theta_{62}^1 & \theta_{62}^2 & \theta_{62}^3 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \alpha & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \rho & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ \alpha\rho & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ \rho\alpha & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ \alpha\rho\alpha & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ \pi & 1 & 2 & 2 & 2 & 3 & 4 & 2 & 4 & 3 \\ \alpha\pi & 1 & 2 & 2 & 2 & 3 & 4 & 2 & 4 & 3 \\ \eta & 2 & 2 & 2 & 2 & 4 & 4 & 2 & 4 & 4 \end{array} \right).$$

Case (7). We have

$$\left(\begin{array}{c|cc|cc} L_{\mathcal{AH}_1}^{(7)} & \theta_{71}^1 \otimes - & \theta_{71}^2 \otimes - & & \\ & \theta_{71}^1 & \theta_{71}^2 & \theta_{71}^1 & \theta_{71}^2 \\ \hline 1 & 1 & 0 & 0 & 1 \\ \chi & 2 & 2 & 2 & 3 \\ \psi & 1 & 2 & 2 & 2 \\ \tau & 4 & 6 & 6 & 7 \\ \zeta & 3 & 4 & 4 & 5 \\ \sigma & 3 & 2 & 2 & 4 \end{array} \right),$$

$$\left(\begin{array}{c|ccc|ccc|ccc} M_{\mathcal{AH}_2}^{(7)} & \theta_{72}^1 \otimes - & & & \theta_{72}^2 \otimes - & & & \theta_{72}^3 \otimes - & & \\ & \theta_{72}^1 & \theta_{72}^2 & \theta_{72}^3 & \theta_{72}^1 & \theta_{72}^2 & \theta_{72}^3 & \theta_{72}^1 & \theta_{72}^2 & \theta_{72}^3 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \alpha & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \rho & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 \\ \alpha\rho & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ \rho\alpha & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ \alpha\rho\alpha & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 \\ \pi & 1 & 2 & 2 & 2 & 4 & 3 & 2 & 3 & 4 \\ \alpha\pi & 1 & 2 & 2 & 2 & 3 & 4 & 2 & 4 & 3 \\ \eta & 2 & 2 & 2 & 2 & 4 & 4 & 2 & 4 & 4 \end{array} \right),$$

$$\left(\begin{array}{c|ccc|ccc|ccc} N_{\mathcal{AH}_3}^{(7)} & \theta_{73}^1 \otimes - & & & \theta_{73}^2 \otimes - & & & \theta_{73}^3 \otimes - & & \\ & \theta_{73}^1 & \theta_{73}^2 & \theta_{73}^3 & \theta_{73}^1 & \theta_{73}^2 & \theta_{73}^3 & \theta_{73}^1 & \theta_{73}^2 & \theta_{73}^3 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \beta & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \xi & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 4 \\ \beta\xi & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 4 \\ \xi\beta & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 4 \\ \beta\xi\beta & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 4 \\ \mu & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 2 & 7 \\ \beta\mu & 0 & 1 & 2 & 1 & 0 & 2 & 2 & 2 & 7 \\ \nu & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 6 \end{array} \right).$$

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