Comparing Graphs of Different Sizes

by Russell Lyons

Abstract. We consider two notions describing how one finite graph may be larger than another. Using them, we prove several theorems for such pairs that compare the number of spanning trees, the return probabilities of random walks, and the number of independent sets, among other combinatorial quantities. Our methods involve inequalities for determinants, for traces of functions of operators, and for entropy.

§1. Introduction.

How does one compare different graphs? If they have the same vertex sets but one edge set contains the other, then clearly we can say that one graph is larger than the other. This can lead to inequalities, usually trivial, for combinatorial quantities associated with the graphs. But if the numbers of vertices differ, then one might wish to compare those combinatorial quantities with normalizations that depend on the numbers of vertices. In such a case, however, if one graph can be embedded in another, we are unlikely to have any such comparison of normalized combinatorial quantities. Instead, we should demand some sort of uniformity in how one graph can be embedded in the other. With appropriate hypotheses of uniformity, we have found inequalities for counting any of the following: spanning trees; independent sets; proper colorings; acyclic orientations; forests; and matchings. We also have inequalities for random walks and the spectra of the graphs. However, many questions remain open.

We now describe what we mean by uniformity of embedding. Let G and H be finite connected (multi)graphs. We will use G for the larger graph. The simplest kind of uniformity is that H ***tiles*** G, meaning that G contains a collection of copies of H that cover each vertex of G exactly once. See Figure 1 for an example. The case where H tiles G is hardly different from H being a subgraph of G with the same number of vertices, and will not be discussed further here.

2000 Mathematics Subject Classification. Primary 05C05, 60C05. Secondary 05C80, 05C81.
Key words and phrases. Spanning trees, random walks, independent sets, matchings.
Research partially supported by NSF grants DMS-0406017 and DMS-1007244, and Microsoft Research.

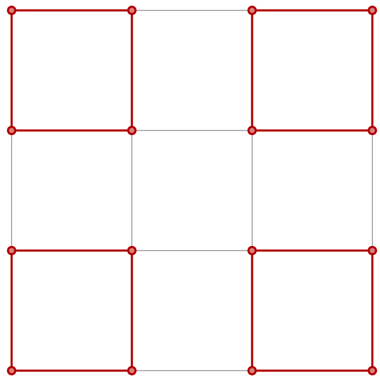

Figure 1. The 4×4 portion of the square grid is tiled by 4 copies of a 4-cycle.

In general, we define a ***copy*** of H to be a subgraph of G that is isomorphic to H. We will also refer to a copy of H as an embedding of H.

The next simplest kind of uniformity is that G has a ***fractional tiling*** by H. This means that there is an integer number of copies of H in G such that each vertex of G is covered the same number of times by these copies of H. An example is in Figure 2. This is already nontrivial and will be a common hypothesis in our paper.

Figure 2. The graph K_4 is fractionally tiled by K_3.

Finally, the most general case we will consider is that G ***dominates*** H, written $G \succcurlyeq H$, meaning that there is a probability measure on pairs $(X, Y) \in V(G) \times V(H)$ such that almost surely there is a rooted isomorphism from (H, Y) to a subgraph of (G, X) and such that the marginal distributions of X and Y are each uniform. Here, a rooted graph is a pair (G, o) with $o \in V(G)$ and a rooted isomorphism is an isomorphism that carries one root to the other. The way to think of domination is that G looks bigger than H from the point of view of a typical vertex. For some illustrative examples, see Figures 3 and 4.

We say that a graph is ***transitive*** if for every pair of vertices, there is an automorphism of the graph that takes one to the other. If H is transitive, then $G \succcurlyeq H$ iff every vertex of G belongs to a copy of H. If G is transitive, then $G \succcurlyeq H$ iff G contains a copy of H. In both cases,

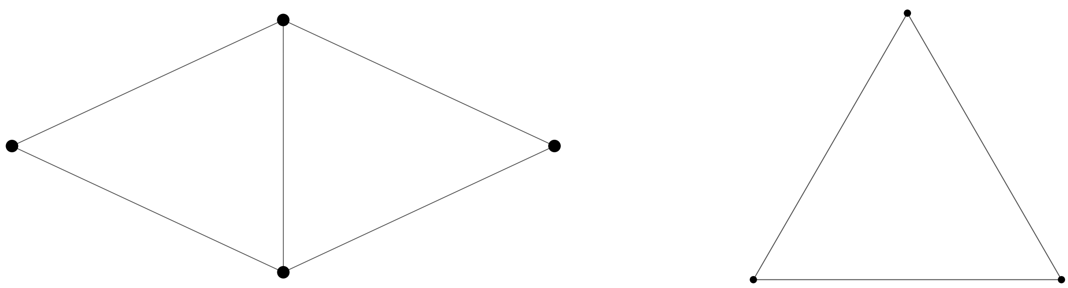

Figure 3. The graph G on the left dominates the graph H on the right, but H does not fractionally tile G.

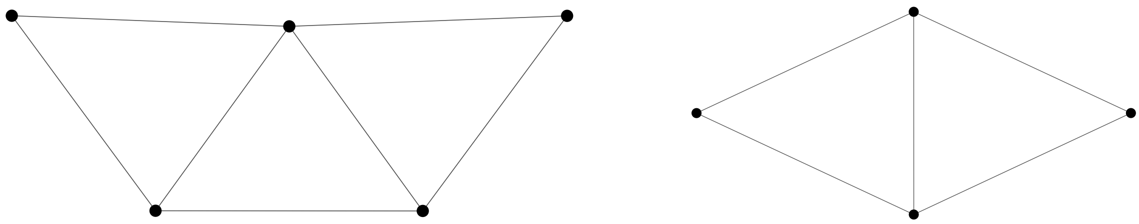

Figure 4. The graph on the left dominates the graph on the right.

the independent coupling of roots works. It is clear that if H fractionally tiles G, then $G \succcurlyeq H$. Conversely, if G is transitive and dominates H, then H fractionally tiles G.

Consider the case where H is a single edge. Then to say that G dominates H is to say that G has no isolated vertices; since G is connected, this means that G has at least two vertices. On the other hand, to say that H fractionally tiles G is to say that there is a spanning "subgraph" of G that is regular of degree at least 1; the reason for the quotes is that the subgraph may need to use edges of G multiple times and thus be a multigraph even if G is a simple graph. See Figure 5 for a comparison.

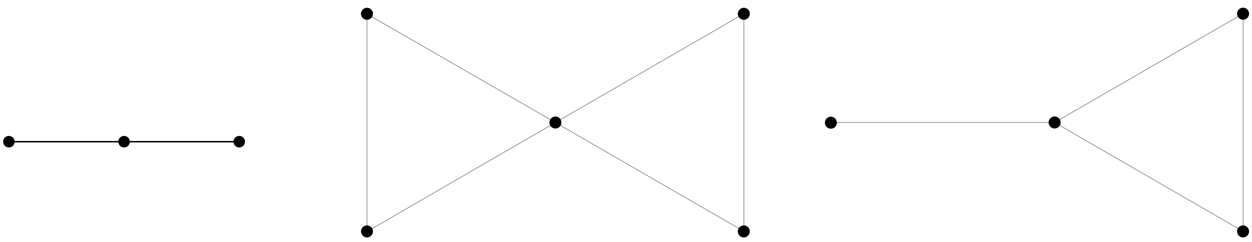

Figure 5. The graph on the left dominates an edge; an edge fractionally tiles the graph in the middle and tiles the graph on the right.

In probabilistic language, $G \succcurlyeq H$ has a simple expression, though we will not use it. Note that the set of (isomorphism classes of) rooted graphs is partially ordered by rooted embedding. This partial order defines a corresponding notion of stochastic ordering $\succcurlyeq$ on the set of probability measures on rooted finite graphs. Let $U(G)$ denote the probability measure on G with a uniformly random root. Then $G \succcurlyeq H$ iff $U(G) \succcurlyeq U(H)$.

Our theme is the following. Suppose we know an inequality of the form $f(H) \le f(G)$ when H is a spanning subgraph of G. Does it extend with appropriate normalization to the setting of domination or fractional tiling?

One might consider weighted graphs as well, but in most cases, we will not. One can also consider random graphs; see Sec. 6 of Lyons, Peled, and Schramm (2008) for several such questions. Luczak and Winkler (2004) and Janson (2006) contain further information for random trees.

If G is a connected graph, we call a subgraph of G a ***spanning tree*** if it is maximal without cycles. The number of spanning trees of G is denoted $\tau(G)$.

We conjecture the following:

CONJECTURE 1.1. *If $G \succcurlyeq H$, then*

$$\tau(G)^{1/|G|} \ge \tau(H)^{1/|H|} . \tag{1.1}$$

The infinitary analogue of Conjecture 1.1 for unimodular probability measures is true, though we will not use it explicitly; see Lyons (2010). We will establish several special cases of Conjecture 1.1. These proofs will use the Hadamard–Fischer–Koteljanskii inequality for determinants, which we review in Section 2. One could extend our considerations to stochastic domination of probability measures of the form $U(G)$ and $U(H)$ for G and H themselves random, but for simplicity, we usually avoid such. The infinitary analogue of Conjecture 1.1 also implies that (1.1) holds for certain pairs of large graphs: see Proposition 3.5 below.

After treating spanning trees, we will give inequalities for return probabilities of continuous-time random walks and for eigenvalues when one graph fractionally tiles another. The main tool here will be a trace inequality for functions of operators.

Our last section presents some easy consequences of Shearer's inequality about entropy for a variety of combinatorial quantities, such as counting the number of independent sets. The last two sections contain several open questions.

§2. Determinant Inequalities.

For a square matrix M and a subset A of the indices of its rows and columns, let $M(A)$ denote the minor of M corresponding to the rows and columns indexed by A. We use the convention $M(\varnothing) := 1$. The Hadamard–Fischer–Koteljanskii inequality says that if M is a positive semidefinite matrix, then

$$M(A)M(B) \ge M(A \cup B)M(A \cap B)\,.$$

In other words, $M(\bullet)$ is log-submodular. It follows that if the index of each row belongs to precisely m sets A_i (each of arbitrary size), then

$$\prod_i M(A_i) \ge (\det M)^m\,. \tag{2.1}$$

Indeed, we simply aggregate repeatedly any pairs of subsets where neither is contained in the other. Each time we get a larger subset and this can only end when we have m copies of the entire index set (together with irrelevant empty sets).

From now on, ***all graphs we consider will be connected without mention***.

One well-known example of a positive semidefinite matrix associated to a finite graph, G, is its ***Laplacian***, Δ_G, whose off-diagonal entries $\Delta_G(x, y)$ are negative the numbers of edges joining x and y and whose row sums each vanish. For $W \subset V(G)$, denote by G/W the graph obtained from G by identifying all vertices in W to a single vertex. By the Matrix-Tree theorem, for any non-empty subset $W \subset V(G)$, we have

$$\tau(G/W) = \Delta_G\big(V(G) \setminus W\big)\,. \tag{2.2}$$

Thus, if we denote $G/(V(G) \setminus A)$ by G_A, then we have

$$\tau(G_A)\tau(G_B) \ge \tau(G_{A\cup B})\tau(G_{A\cap B}) \tag{2.3}$$

when $A \cup B$ is a proper subset of $V(G)$.

Inequality (2.3) does not hold when $A \cup B = V(G)$. For example, take G to be a path on 3 vertices, x, y, and z, with y the middle vertex. Let $A := \{x, y\}$ and $B := \{y, z\}$. Then the left-hand side of (2.3) is 1 while the right-hand side is 2. However, (2.3) does hold when $A \cup B = V(G)$ and there is an edge between $A \setminus B$ and $B \setminus A$. Indeed, such an edge may be subdivided to create a new vertex x that does not belong to either A or B. Let the new graph be G' with vertex set $A \cup B \cup \{x\}$. Note that $\tau(G'_A) = \tau(G_A)$, $\tau(G'_B) = \tau(G_B)$, $\tau(G'_{A\cup B}) = 2 > 1 = \tau(G_{A\cup B})$, and $\tau(G'_{A\cap B}) \ge \tau(G_{A\cap B})$. Thus, if we apply (2.3) to G' with these same A and B, we obtain an inequality that is stronger than (2.3) applied to G, as desired.

As Jeff Kahn noted (personal communication), this extension of (2.3) implies the following inequality: if $A_i \subset V(G)$ and each $x \in V(G)$ belongs to exactly m sets A_i, then

$$\prod_i \tau(G_{A_i}) \geq \tau(G)^m\,, \tag{2.4}$$

since if $A_i \neq V(G)$, then there is an $x \in A_i$ adjacent to some $y \notin A_i$, whence there is some A_j containing y but not x, as x is covered the same number of times as y is. Hence we can apply (2.3) or its extension repeatedly.

Some final notation for a (possibly disconnected) subgraph H of G: Write G_H for $G_{V(H)}$. Write $G/\!/H := G/E(H)$, the graph obtained from G by contracting all edges in $E(H)$. When H is connected, $G/\!/H = G/V(H)$. By $|G|$, we mean $|V(G)|$.

§3. Spanning Trees.

In this section, we prove (1.1) when $G \succcurlyeq H$ and either G or H is transitive, and in general when G is fractionally tiled by H.

Lemma 3.1. *For any G and $H \subseteq G$, we have*

$$\tau(H)\tau(G/\!/H) \leq \tau(G)\,.$$

Proof. The union of the edges of a spanning tree of H and a spanning tree of $G/\!/H$ is a spanning tree of G. This map from the pairs of spanning trees of H and the spanning trees of $G/\!/H$ to the spanning trees of G is obviously injective. ∎

Lemma 3.2. *If G is transitive and $H \subseteq G$, then*

$$\tau(G_H) \geq \tau(G)^{|H|/|G|}\,.$$

Proof. We prove this by induction on $|H|$. We may assume that $\varnothing \neq H \neq G$. Call an image of H under an automorphism of G a ***clone*** of H. We claim that there is an edge e such that we can cover $V(G)$ by clones of H, none of which uses both endpoints of the edge e. Indeed, cover by clones as much of $V(G)$ as possible without covering all of $V(G)$. Let o be a vertex that is not covered. Any clone of H that covers o covers all other uncovered vertices of G. Further, some vertex $x \in V(H)$ has degree less than its degree in G. Choose an automorphism of G that maps x to o and use the corresponding clone of H to finish the cover of $V(G)$. Let o' be a neighbor of o that is not contained in this last clone of H. The edge $e = (o, o')$ is the edge we desire. Let k be the total number of clones H_i of H used in this cover of $V(G)$.

Now let G' be G subdivided at e by a new vertex, z. We have $V(G' \setminus z) = \bigcup_{i=1}^{k} V(H_i)$. Note that for each i, G'_{H_i} is isomorphic to G_{H_i}, possibly plus a loop, because H_i does not contain both o and o'. Furthermore, G_{H_i} is isomorphic to G_H by definition of clone. We may assume that each H_i has a vertex not belonging to $L_i := \bigcup_{j<i} H_j$. By (2.3), we have

$$\tau(G'_{L_i})\tau(G'_{H_i}) \geq \tau(G'_{L_{i+1}})\tau(G'_{K_i}),$$

where $K_i := L_i \cap H_i \subsetneq H_i$. Since $V(L_{k+1}) = V(G' \setminus z)$, we have $G'_{L_{k+1}} = G'$. In addition, since $K_i \subset H_i$, we have G'_{K_i} is G_{K_i}, possibly plus a loop. Thus, multiplying together the above inequalities for $1 \leq i \leq k$ and cancelling common terms on both sides yields

$$\tau(G_H)^k = \prod_{i \geq 1} \tau(G'_{H_i}) \geq \tau(G') \prod_{i>1} \tau(G'_{K_i}) = \tau(G') \prod_{i>1} \tau(G_{K_i}) \geq \tau(G) \prod_{i>1} \tau(G_{K_i}).$$

Since $|K_i| < |H_i|$, the inductive hypothesis gives $\tau(G_{K_i}) \geq \tau(G)^{|K_i|/|G|}$. Now $k|H| = |G| + \sum_{i>1} |K_i|$, whence

$$\tau(G_H)^k \geq \tau(G)^{k|H|/|G|},$$

which is the desired inequality. ∎

Strict inequality holds in Lemma 3.2 when $|G| > |H| \geq 1$ and G contains no cut-edge, since in that case, $\tau(G') > \tau(G)$ in the proof.

Theorem 3.3. *If G is transitive, then (1.1) holds with strict inequality when G contains no cut-edge and $G \neq H$.*

Proof. Let K be the complement of the vertices in a copy of H in G. The previous two lemmas give

$$\tau(G)^{|K|/|G|} \leq \tau(G_K) = \tau(G//H) \leq \frac{\tau(G)}{\tau(H)}.$$

Since $|K| + |H| = |G|$, the desired inequality follows. ∎

We now prove that Conjecture 1.1 holds when H is transitive.

Theorem 3.4. *If H is transitive, then (1.1) holds.*

Proof. We prove this by induction on $|G|$. If $|G| = |H|$, then G contains a copy of H and the result is trivial. Otherwise, cover $V(G)$ by copies H_i of H for $1 \leq i \leq k$. For $1 < i < k$, we may assume that H_i has a vertex either in or adjacent to H_j for some $j < i$. Let $V' := \bigcup_{1 \leq i < k} V(H_i)$ and G' be the graph spanned by V'. Then G' is connected. We may also assume that $H' := H_k$ has a vertex not in V'. Since H is transitive and G' is covered by copies of H, we know that $G' \succcurlyeq H$, whence our inductive hypothesis says that

$$\tau(G')^{1/|G'|} \geq \tau(H)^{1/|H|}.$$

If H' does not contain a vertex in V', then

$$\tau(G) \geq \tau(G')\tau(H') \geq \tau(H)^{|G'|/|H|}\tau(H') = \tau(H)^{|G|/|H|}\,.$$

If H' does contain a vertex in V', then note that $G//G' = G/V'$ is isomorphic, up to loops, to $H'/(V' \cap H') = H'_{H'\setminus V'}$. Thus, the previous two lemmas give

$$\begin{aligned}\tau(G) &\geq \tau(G')\tau(G//G') \geq \tau(H)^{|G'|/|H|}\tau(H'_{H'\setminus V'}) \\ &\geq \tau(H)^{|G'|/|H|}\tau(H)^{|H'\setminus V'|/|H|} = \tau(H)^{|G|/|H|}\,.\end{aligned}$$

Both cases together complete the induction. ∎

Note that the same proof shows that if H is any graph such that every (connected) subgraph $K \subset H$ satisfies $\tau(H_K) \geq \tau(H)^{|K|/|H|}$, then for every G each of whose vertices belongs to a copy of H [in particular, if $G \succcurlyeq H$], we have $\tau(G)^{1/|G|} \geq \tau(H)^{1/|H|}$. Many small non-transitive graphs H can be shown to have this property.

We also show that pairs of large graphs tend to satisfy Conjecture 1.1. Write $\|\bullet\|$ for the usual total-variation norm of signed measures. Also, write $U_r(G)$ for the distribution of (the isomorphism class of) the rooted ball of radius r about a uniform random root of G.

Proposition 3.5. *Suppose that $D, r < \infty$ and $\epsilon > 0$. There is some $k < \infty$ such that if $G \succcurlyeq H$, $|H| \geq k$, all degrees in G are at most D, and $\|U_r(G) - U_r(H)\| \geq \epsilon$, then $\tau(G)^{1/|G|} \geq \tau(H)^{1/|H|}$.*

Proof. If not, then there is a sequence $G_n \succcurlyeq H_n$ with $|H_n| \to \infty$, all degrees of G_n are at most D, $\|U_r(G_n) - U_r(H_n)\| \geq \epsilon$, and $\tau(G_n)^{1/|G_n|} < \tau(H_n)^{1/|H_n|}$. By compactness, there is a subsequence, which for simplicity of notation we take to be the whole sequence, such that $U(G_n)$ weakly converges to some probability measure μ on rooted graphs and $U(H_n)$ weakly converges to some probability measure ν on rooted graphs. Then $\|\mu - \nu\| \geq \epsilon$. By Theorem 3.2 of Lyons (2005) and Theorem 3.3 of Lyons (2010), we have that $\lim_{n\to\infty} \tau(G_n)^{1/|G_n|} > \lim_{n\to\infty} \tau(H_n)^{1/|H_n|}$, a contradiction. ∎

We remark that weaker assumptions suffice in place of the bounded degree assumption; as long as tightness and bounded average log degree hold for a class of graphs, the same argument works. See Section 3 of Benjamini, Lyons, and Schramm (2015) for a discussion of tightness.

We owe the following result to Jeff Kahn.

Theorem 3.6. *If H fractionally tiles G, then (1.1) holds.*

Proof. Let H_i be the copies of H that fractionally tile G and $A_i := V(G) \setminus V(H_i)$ for $1 \leq i \leq m$. We have by Lemma 3.1 that

$$\tau(G)^m \geq \prod_i \tau(H_i)\tau(G//H_i)\,.$$

Since H_i is connected, we have $G//H_i = G_{A_i}$, so now we can apply (2.4). Each vertex of G appears $m\big(|G| - |H|\big)/|G|$ times in some A_i, whence

$$\tau(G)^m \geq \tau(G)^{m(|G|-|H|)/|G|} \prod_i \tau(H_i) = \tau(H)^m \tau(G)^{m(|G|-|H|)/|G|} .$$

This gives the desired inequality. ∎

In summary, we have proved that Conjecture 1.1 holds under any of the following additional hypotheses: if either G or H is transitive; if H fractionally tiles G; if H is any graph such that every (connected) subgraph $K \subset H$ satisfies $\tau(H_K) \geq \tau(H)^{|K|/|H|}$; if H is sufficiently large and G and H are sufficiently distinct (see Proposition 3.5 for details).

§4. Fractional Tiling and Random Walks.

For a continuous-time random walk on a weighted simple graph G, let $p_t(x; G)$ denote the probability that a random walk started at x is at x at time t. If Δ_G is the corresponding Laplacian, i.e., $\Delta_G(x, y) := -w(e)$ when e is an edge joining x and y with weight $w(e)$, all other off-diagonal elements of Δ_G are 0, and the row sums are 0, then $p_t(x; G)$ is the (x, x)-entry of $e^{-t\Delta_G}$.

We would like to prove that if G dominates H, then for all $t > 0$,

$$\frac{1}{|G|} \sum_{x \in V(G)} p_t(x; G) \leq \frac{1}{|H|} \sum_{x \in V(H)} p_t(x; H) . \tag{4.1}$$

It is easy to see that this inequality holds near 0 and near ∞. One motivation is the following open problem of Fontes and Mathieu (personal communication). Suppose that G is a fixed Cayley graph and w_1, w_2 are two random fields of positive weights on its edges with the following properties: Each field w_i has an invariant law and a.s. $w_1(e) \geq w_2(e)$ for each edge e. Does it follow that $\mathbf{E}\big[p_{1,t}(o; G)\big] \leq \mathbf{E}\big[p_{2,t}(o, G)\big]$ for all $t > 0$, where $p_{i,t}$ denotes the return probability to a fixed vertex o at time t with the weights w_i? This is known to be true for amenable G (Fontes and Mathieu, 2006) and also when the pair (w_1, w_2) is invariant (Aldous and Lyons, 2007). The problems for finite graphs and for infinite Cayley graphs are quite similar in that both try to compare different normalized traces.

We prove a partial result, namely, that (4.1) holds when H fractionally tiles G.

THEOREM 4.1. *If G is fractionally tiled by H, then for continuous-time simple random walk, we have for all $t > 0$,*

$$\frac{1}{|G|} \sum_{x \in V(G)} p_t(x; G) \leq \frac{1}{|H|} \sum_{x \in V(H)} p_t(x; H) .$$

Equality holds iff $G = H$.

In fact, a somewhat weaker condition suffices: a number of different graphs can be used inside G as long as their average is at most G in a certain sense, as we formalize next. The equality condition of Theorem 4.1 arises from the proof of Theorem 4.2: we have strict inequality in (4.3) if $G \neq H$. In the following result, the case when $k = m = 1$ is due to Benjamini and Schramm; see Theorem 3.1 of Heicklen and Hoffman (2005).

THEOREM 4.2. *Let G be a graph with positive weights w on its edges. Suppose that H_i is a subgraph of G with positive weights w_i on its edges for $i = 1, \ldots, k$ with the following two properties:*

(i) *there is a constant m such that for every $x \in V(G)$,*

$$\bigl|\{i\,;\ x \in V(H_i)\}\bigr| = m$$

and

(ii) *for every $e \in E(G)$,*

$$w(e) \geq \frac{1}{m} \sum_{i\,;\ e \in E(H_i)} w_i(e)\,.$$

Then for all $t > 0$, we have

$$\frac{1}{|G|} \sum_{x \in V(G)} p_t(x; G) \leq \frac{1}{\sum_{j=1}^{k} |H_j|} \sum_{i=1}^{k} \sum_{x \in V(H_i)} p_t(x; H_i)\,.$$

We will use the notation $A \leq B$ for self-adjoint operators A and B to mean that $B - A$ is positive semidefinite. Sometimes we regard the edges of a graph as oriented, where we choose one orientation (arbitrarily) for each edge. In particular, we do this whenever we consider the ℓ^2-space of the edge set of a graph. In this case, we denote the tail and the head of e by e^- and e^+. Define $d_G\colon \ell^2\bigl(V(G)\bigr) \to \ell^2\bigl(E(G)\bigr)$ by

$$d_G(a)(e) := \sqrt{w(e)}\,\bigl[a(e^-) - a(e^+)\bigr]\,.$$

Then $\Delta_G = d_G^* d_G$. Let Tr denote normalized trace of a square matrix, i.e., the average of the diagonal entries. We use tr for the usual trace.

Proof. Let $n := |G|$ and $N := \sum_{j=1}^{k} |H_j| = nm$. Write $V := V(G)$. Let

$$W := \bigcup_{i=1}^{k} V(H_i) \times \{i\}\,,$$

so that $|W| = N$. Suppose that $\Phi\colon \mathcal{L}\big(\ell^2(W)\big) \to \mathcal{L}\big(\ell^2(V)\big)$ is a positive unital linear map, i.e., a linear map that takes positive operators to positive operators and takes the identity map to the identity map. (Here, a positive operator means positive semidefinite.) Theorem 3.9 of Antezana, Massey, and Stojanoff (2007) says that

$$\operatorname{Tr} f\big(\Phi(A)\big) \le \operatorname{Tr} \Phi\big(f(A)\big) \tag{4.2}$$

for self-adjoint operators $A \in \mathcal{L}\big(\ell^2(W)\big)$ and functions $f\colon \mathbb{R} \to \mathbb{R}$ that are convex on the convex hull of the spectrum of A. (In fact, those authors show the more general inequality $\operatorname{Tr} g\big(f\big(\Phi(A)\big)\big) \le \operatorname{Tr} g\big(\Phi\big(f(A)\big)\big)$ for every increasing convex g.)

We apply this as follows. Write

$$\Gamma(x) := \{i\,;\ x \in H_i\}$$

and define $\phi\colon \ell^2(V) \to \ell^2(W)$ by linearity and the requirement that

$$\phi(\mathbf{1}_{\{x\}}) := \frac{1}{\sqrt{m}} \sum_{i \in \Gamma(x)} \mathbf{1}_{\{(x,i)\}}\,.$$

Then $\phi^*\phi$ is the identity map by hypothesis (i). Define $\Phi\colon \mathcal{L}\big(\ell^2(W)\big) \to \mathcal{L}\big(\ell^2(V)\big)$ by $\Phi T := \phi^* T \phi$. Then Φ is a positive unital map. Regard $H_i \times \{i\}$ as a graph with weights w_i and a corresponding Laplacian matrix Δ_i. Consider the following map $A \in \mathcal{L}\big(\ell^2(W)\big)$:

$$A := \bigoplus_{i=1}^{k} \Delta_i\,.$$

Hypothesis (ii) guarantees that

$$\Delta_G \ge \Phi(A)\,. \tag{4.3}$$

To see this, let $b \in \ell^2(V)$. We have

$$\big(\Delta_G(b),\, b\big) = \|d_G b\|^2 \tag{4.4}$$

and

$$\big(\Phi A(b),\, b\big) = \big(\phi^* A \phi b,\, b\big) = \big(A\phi b,\, \phi b\big)\,. \tag{4.5}$$

Write b_i for the orthogonal projection of ϕb to $\ell^2\big(V(H_i) \times \{i\}\big)$, so that $\phi b = \sum_{i=1}^k b_i$. Note that $b_i(x,i) = b(x)/\sqrt{m}$ for $x \in V(H_i)$. Thus, we have

$$\begin{aligned}\big(A\phi b,\, \phi b\big) &= \sum_{i=1}^{k} \big(\Delta_i b_i,\, b_i\big) = \sum_{i=1}^{k} \|d_{H_i} b_i\|^2 = \frac{1}{m} \sum_{i=1}^{k} \sum_{e \in E(H_i)} \frac{w_i(e)}{w(e)} \big|d_G b(e)\big|^2 \\ &\le \sum_{e \in E(G)} \big|d_G b(e)\big|^2 = \|d_G b\|^2\end{aligned}$$

by hypothesis. Combining this with (4.4) and (4.5), we get our claimed inequality (4.3).

Since (4.3) implies, by the minimax principle, that the eigenvalues of Δ_G are at least the corresponding eigenvalues of $\Phi(A)$, we have

$$\operatorname{Tr} f(\Delta_G) \le \operatorname{Tr} f\big(\Phi(A)\big)$$

for every decreasing function f. (We have strict inequality if f is strictly decreasing and we have strict inequality in (4.3).) Use $f(s) := e^{-ts}$ in this and in (4.2) to obtain

$$\operatorname{Tr} f(\Delta_G) \le \operatorname{Tr} \Phi\big(f(A)\big) . \tag{4.6}$$

The left-hand side equals

$$\frac{1}{n} \sum_{x \in V(G)} p_t(x; G) .$$

We claim that the right-hand side equals

$$\frac{1}{N} \sum_{i=1}^{k} \sum_{x \in V(H_i)} p_t(x; H_i) ,$$

which will complete the proof of the theorem.

Another way to state our claim is that

$$\operatorname{Tr} \Phi\big(f(A)\big) = \frac{1}{N} \sum_{i=1}^{k} \operatorname{tr} f(\Delta_i) .$$

Now, since $f(A) = \bigoplus_{i=1}^{k} f(\Delta_i)$, we have

$$\begin{aligned}
\operatorname{Tr} \Phi\big(f(A)\big) &= \frac{1}{n} \sum_{x \in V} \big(f(A)\phi \mathbf{1}_{\{x\}}, \phi \mathbf{1}_{\{x\}}\big) \\
&= \frac{1}{N} \sum_{x \in V} \sum_{i \in \Gamma(x)} \sum_{j \in \Gamma(x)} \big(f(A) \mathbf{1}_{\{(x,i)\}}, \mathbf{1}_{\{(x,j)\}}\big) \\
&= \frac{1}{N} \sum_{x \in V} \sum_{i \in \Gamma(x)} \sum_{j \in \Gamma(x)} \big(f(\Delta_i) \mathbf{1}_{\{(x,i)\}}, \mathbf{1}_{\{(x,j)\}}\big) \\
&= \frac{1}{N} \sum_{x \in V} \sum_{i \in \Gamma(x)} \big(f(\Delta_i) \mathbf{1}_{\{(x,i)\}}, \mathbf{1}_{\{(x,i)\}}\big) \\
&= \frac{1}{N} \sum_{i=1}^{k} \operatorname{tr} f(\Delta_i) .
\end{aligned}$$

∎

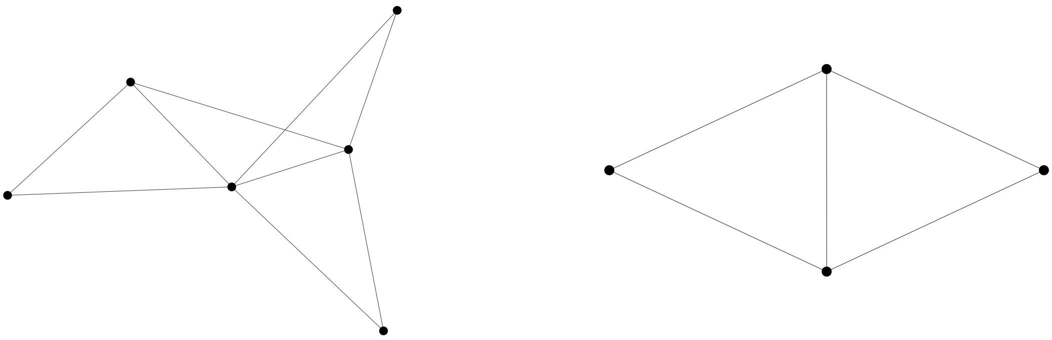

Figure 6. The graph G on the left dominates the graph H on the right.

A similar proof shows that if f is any decreasing convex function and H fractionally tiles G, then

$$\operatorname{Tr} f(\Delta_G) \le \operatorname{Tr} f(\Delta_H) \,. \tag{4.7}$$

However, it is *not* true that this inequality holds whenever $G \succcurlyeq H$; a counter-example is provided by taking $f(t) := (4-t)^+$ and G, H the graphs shown in Figure 6. Possibly, however, it holds whenever $G \succcurlyeq H$ and H is transitive; this is not hard to verify when H is an edge.

Remark 4.3. Theorem 4.2 is sharp in the following sense. If the inequality in (ii) holds in the opposite direction for all edges with strict inequality at least once, then the conclusion fails for all t sufficiently close to 0. This is because both sides equal 1 for $t = 0$, whereas the derivative of the left-hand side at $t = 0$ is

$$-\operatorname{Tr} \Delta_G = -\frac{2}{|G|} \sum_{e \in G} w(e) = -\frac{2 \sum_{e \in G} m w(e)}{\sum_i |H_i|}$$

and the derivative of the right-hand side at $t = 0$ is

$$-\frac{2 \sum_i \sum_{e \in H_i} w_i(e)}{\sum_i |H_i|} \,.$$

For special functions f, we can establish that domination is sufficient for (4.7). A continuous function $f\colon (0,\infty) \to \mathbb{R}$ is called ***operator monotone on*** $(0,\infty)$ if for any bounded self-adjoint operators A, B with spectrum in $(0,\infty)$ and $A \le B$, we have $f(A) \le f(B)$. For example, Löwner (1934) proved that the logarithm is an operator monotone function on $(0,\infty)$ (see also Chapter V of Bhatia (1997)).

Proposition 4.4. *If f is any operator monotone increasing function on $(0,\infty)$ and G dominates H, then*

$$\operatorname{Tr} f(\Delta_G + t) \ge \operatorname{Tr} f(\Delta_H + t) \tag{4.8}$$

and

$$\det(\Delta_G + tI)^{1/|G|} \ge \det(\Delta_H + tI)^{1/|H|} \tag{4.9}$$

for $t > 0$.

Proof. Fix $t > 0$ and define $g(s) := f(s+t)$. Consider a copy K of H in G and some vertex $x \in V(K)$. We have $\Delta_G \ge \Delta_K \oplus \mathbf{0}$. Therefore $g(\Delta_G) \ge g(\Delta_K \oplus \mathbf{0}) = g(\Delta_K) \oplus g(0)I$. Comparing the (x,x)-entries, we obtain $g(\Delta_G)(x,x) \ge g(\Delta_K)(x,x)$.

The definition of $G \succcurlyeq H$ is that there is a certain coupling of copies of (H,Y) and (G,X) with Y mapping to X; for each such copy (K,X), apply the preceding inequality and take expectation. This gives (4.8).

Taking $f = \log$ yields (4.9). ∎

§5. Fractional Tiling and Independent Sets.

There are some easy results that follow from Shearer's inequality (Chung, Graham, Frankl, and Shearer, 1986), which states the following:

THEOREM 5.1. *Given discrete random variables* $X_1, \ldots, X_k$ *and* $S \subseteq [1,k]$, *write* X_S *for the random variable* $\langle X_i\,;\ i \in S\rangle$. *Let* $\mathscr{S}$ *be a collection of subsets of* $[1,k]$ *such that each integer in* $[1,k]$ *appears in exactly* r *of the sets in* $\mathscr{S}$. *Then*

$$r\,\mathbf{H}(X_1, \ldots, X_k) \le \sum_{S \in \mathscr{S}} \mathbf{H}(X_S)\,.$$

Here, $\mathbf{H}(X) := -\sum_x \mathbf{P}[X = x] \log \mathbf{P}[X = x]$ denotes the entropy of a discrete random variable X.

A set of vertices in a graph is ***independent*** if no pair in the set is adjacent. A ***homomorphism*** from G to H is a map from $V(G)$ to $V(H)$ that sends adjacent vertices to adjacent vertices. If $w\colon V(H) \to (0,\infty)$ is a weight function, then the ***weight*** of a map $\phi\colon V(G) \to V(H)$ is $\prod_{x \in V(G)} w(\phi(x))$. The total weight of a set of such maps is just the sum of the weights of the individual maps.

PROPOSITION 5.2. *Let* $f(G)$ *denote one of the following:*

- *the number of independent sets in* G*;*
- *the number of proper colorings of* G *with a fixed number of colors;*
- *the total weight of the homomorphisms of* G *to a fixed graph* F *with arbitrary positive weights on the vertices of* F.

If H fractionally tiles G, then

$$f(G)^{1/|G|} \le f(H)^{1/|H|} \,. \tag{5.1}$$

Proof. For each of the things we count, the restriction of one of them in G to a copy of H is also one of them for H. Thus, (5.1) is immediate from Shearer's inequality: For example, if S is a random uniform independent set in G, then its entropy $h(S)$ equals $\log f(G)$. Let $Z_x := \mathbf{1}_{\{x \in S\}}$. Let H_j $(j \in J)$ be the copies of H that fractionally tile G. Each vertex of G belongs to exactly $|J| \cdot |H|/|G|$ of these copies of H. Since the restriction of S to $V(H)$ is an independent set in H, we have

$$\begin{aligned}\log f(G) &= h\big(\langle Z_x\,;\; x \in V(G)\rangle\big)\\ &\le \frac{|G|}{|J|\cdot|H|}\sum_j h\big(\langle Z_x\,;\; x \in V(H_j)\rangle\big)\\ &\le \frac{|G|}{|J|\cdot|H|}\sum_j \log f(H_j) = \frac{|G|}{|H|}\log f(H)\end{aligned}$$

by Shearer's inequality.

For weighted homomorphisms, standard techniques apply: it suffices to prove it for rational weights or, by homogeneity, for integral weights. Then we blow up each vertex of F a certain number of times to get an equivalent inequality for an unweighted graph, F', which follows as above. (Here, F' has vertex set $\big\{(x,i)\,;\; x \in V(F),\, 1 \le i \le w(x)\big\}$ and an edge from (x,i) to (y,j) whenever $(x,y) \in E(F)$.) ∎

A similar inequality holds for fractional tilings by varied graphs, rather than by a fixed graph. That is, if G is fractionally tiled by $H_1, \ldots, H_k$, meaning that each H_i is a subgraph of G and each vertex of G belongs to the same number of H_i, then

$$f(G)^{1/|G|} \le \Big(\prod_{j=1}^{k} f(H_j)\Big)^{1/\sum_{j=1}^{k}|H_j|} \,.$$

Of course, similar inequalities hold for hypergraphs.

We do not know when (5.1) holds under the weaker assumption that G dominates H. It does not hold when f counts independent sets, as the example of G being a star and H being an edge shows. For proper colorings, however, it is easy to check that this inequality does hold when H is an edge.

The following is proved similarly to Proposition 5.2, but with basic random variables representing edges rather than vertices. In this proposition, we say that H ***fractionally edge-tiles*** G if there is a set of copies of H in G such that each edge of G belongs to the same number of copies of H in the set.

Proposition 5.3. *Let $f(G)$ denote one of the following:*

- *the number of acyclic orientations of G (this is an evaluation of the Tutte polynomial, $T_G(2,0)$);*
- *the number of forests in G (this is an evaluation of the Tutte polynomial, $T_G(2,1)$);*
- *the number of matchings in G.*

If H fractionally edge-tiles G, then

$$f(G)^{1/|E(G)|} \le f(H)^{1/|E(H)|}. \tag{5.2}$$

We do not know when the inequality opposite to (5.1) holds under the weaker assumption that G dominates H for the functions f of Proposition 5.3. It does not hold when f counts matchings, as the example of G being a star and H being an edge shows. It might be the case that for any $x, y \ge 1$, we have $T_G(x,y)^{1/|G|} \ge T_H(x,y)^{1/|H|}$ when G dominates H, or even that all the coefficients of $T_G(x+1,y+1)^{|H|} - T_H(x+1,y+1)^{|G|}$ are non-negative. Random testing of pairs $G \succcurlyeq H$ supports the possibility that all coefficients are non-negative in this difference. Of course, such an inequality would imply (1.1). When H is a tree, it is easy to prove this inequality.

We close with a few questions involving fractional tiling.

Let $f(G)$ be the number of matchings of G. Is

$$f(G)^{1/|G|} \ge f(H)^{1/|H|} \tag{5.3}$$

when H fractionally tiles G? After discussions with Ádám Timár, he proved this holds when H is an edge. To see this, consider a maximal matching, M, of G. Let $W := V(G) \setminus V(M)$. By definition, W is an independent set. In a fractional tiling of G by H, let K be the list of all the copies of H that use a vertex of W. This may include repetitions. The fact that H fractionally tiles G combined with Hall's marriage theorem allows us to conclude that there is a matching M' that contains W with M' using only edges from K, and where each edge in M' intersects W. Now $M \cup M'$ is a subgraph of G that satisfies the inequality, as can be seen by considering the connected components of $M \cup M'$. That is,

$$f(G)^{1/|G|} \ge f(M \cup M')^{1/|G|} \ge 2^{1/2} = f(H)^{1/|H|}.$$

More generally, call a set of subgraphs of G a ***packing*** if the subgraphs are disjoint. Let $f(G)$ be the number of packings of G by copies of a fixed graph K (so when K is an edge, this is the number of matchings). Does (5.3) hold when H fractionally tiles G? What about the particular case $K = H$?

Acknowledgements. I am grateful to Oded Schramm for discussions at the start of this project and to Prasad Tetali for conversations about Section 5. I thank Jeff Kahn and Ádám Timár for permission to include their proofs.

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