THE EQUATIONS OF THE MULTI-PHASE HUMID ATMOSPHERE EXPRESSED AS A QUASI VARIATIONAL INEQUALITY

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Abstract. In this article we propose a new formulation of the equations of the humid atmosphere with a multi-phase saturation generalizing thus the model introduced in [TWu15] and [TWa15]. More precisely, we consider the more realistic situation where the humid atmosphere comprises these components, namely water vapor, liquid water and cloud condensates and furthermore the saturation concentration is not constant. With the additional constraint that the vapor mass ratio q_v is less than the saturation concentration q_{vs}, which depends itself on the state, we are led, from the mathematical point of view, to introduce and handle a system of equations and inequations involving some *quasi-variational inequalities* for which we prove the existence of solutions.

1. Introduction

A phenomenon as common as the clouds is nevertheless far to be understood from the physical point of view and the specialists believe that the clouds (and the aerosols participating in their formation and evolution) is the *greatest source of uncertainty* regarding the current numerical simulations for weather and climate predictions.

Clouds are made of many components, air, water, liquid water, ice, pollutants, etc.

The mathematical theory of the equations of the humid atmosphere [Gil82], [Ped87] has been initiated in [LTW92] and more recently in [GH06, GH11]. However, in these references, the humidity is only accounted for through the mass fraction q of air vapor; in addition the saturation of water vapor in the air is not accounted for, so that the equation for the concentration q of water vapor in the air is a mere transport equation. To the best of our knowledge the first articles accounting for the water saturation are [CT12], [CFTT13], and [BCT14]. In these articles the existence of a change of phase leads to the introduction of a Heaviside function, so that the equations for q and T (the temperature) appear as nonlinear, discontinuous and non-monotone. Nevertheless results of existence, uniqueness, maximum principle and regularity of solutions were established. For other equations involving a discontinuous Heaviside function in geophysics see e.g. [Dia93, DT99], and [Fei91, FN91, Gil82] in more general contexts.

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Two simplifying assumptions were made in the references [BCT14, CFTT13, CT12]: namely that the velocity of the fluid $\mathbf{u}$ is known and that the saturation concentration q_{vs} is constant. In [CHKTZ], the authors removed the hypothesis that the velocity is known and they studied the coupled system for $q, T, \mathbf{u}$, thus combining the methods in [CFTT13, CT12] with the methods for the 3-dimensional primitive equations (PEs), [CT07], [Kob06].

In the article [TWu15] the authors assumed again that the velocity $\mathbf{u}$ is prescribed but they also assumed, for simplicity, that the saturation concentration q_{vs} is constant. They then observed that the basic equations for q and T as reported e.g. in the classical references [Hal71, HW80, RY89] are inconsistent for the limit values $q = 0$ and $q = 1$. This difficulty was reported in the geophysics literature in [TT16], and, in [TWu15] and [TT16] the authors propose to resolve the contradiction in the equations by introducing a unilateral equation (an inequality) valid for $q = 0$ and $q = 1$, and then the whole problem is set as a *variational inequality.* For general results on variational inequalities and their utilization in mechanics and physics, see e.g. [Bre72, DL76, Fre02, KS80, ET76].

In the present article, we generalize the work of [TWu15] (see also [TWa15]) by considering a more detailed description of the humid atmosphere, namely we assume that the humid quantities comprise the water vapor, the cloud-condensates, and rain water with respective mass densities q_v, q_c and q_r. In the earlier works ([TWu15], [TWa15] and before) q_v is the quantity which was called q (and q_{vs} was called q_{vs}) . Because of the increased complexity of the model we first recall in Section 2 all the equations, mostly based on the references [Gra98, Hal71, KW78, KBH98, RY89, Xue89] in view also of setting the notations, and putting the equations in a form suitable for mathematical treatment. The mathematical treatment of the problem is conducted in Sections 3 and 4.

The multi-species model that we consider in this article is described below. Meanwhile, a different model for multi-species humid atmosphere was introduced in [KM06] and studied from the mathematical viewpoint in the recent article [HKLT17]. As explained in [KM06] (see after (9) in [KM06]), it is often assumed in cloud microphysics parameterizations that the vapor-to-cloud water conversion is instantaneous, i.e. that *either* the air is saturated, such that the water vapor content matches its saturation value, $q_v = q_{vs}(T, p)$, and the cloud water droplets can exist with $q_c > 0$, *or* the air is undersaturated , i.e. $q_v < q_{vs}$, in which case $q_c \equiv 0$. See [Gra98] and other references below; this is our point of view here. In [KM06] and [HKLT17], the authors do not assume this limiting behavior from the outset and demonstrate how it may be derived in a consistent asymptotic framework given large but finite condensation rates. This is the main deviation of the bulk microphysics description in [KM06] from the scheme related to [Gra98] that we study here.

2. Mathematical formulation of the problem

Following the references quoted above above, we consider the conservation equations for the relative mass densities $q_v,\ q_c,\ q_r$ and for the temperature T (or more precisely the difference θ' between the potential temperature θ and a reference temperature θ_h, $\theta' = \theta - \theta_h$).

We let $\mathcal{M} \subset \mathbb{R}^3$ be the spatial domain for our study and a typical point in $\mathcal{M}$ is denoted by $\mathbf{x} = (x, y, p)$ where p is the pressure. We use ρ, q, θ, T and e_{vs} to denote density, concentration, potential temperature, temperature, and saturation vapor pressure, respectively. In earlier works [BCT14], [TWu15], [TWa15], we considered the atmosphere to be a mixture of dry air and water vapor. In our current investigation, we will consider the water vapor, cloud-condensate and rain water for the humid atmosphere so as to include the clouds. For a specific quantity, we shall use the subindices v, c, and r to represent this quantity for the water vapor, cloud-condensate and rain water. For example, ρ_v, q_v represent the density and concentration of water vapor, ρ_c, q_c the density and concentration of cloud-condensate, and ρ_r, q_r the density and concentration of rain water, etc.

Assuming the velocity $\mathbf{u} = (u, v, \omega)$ is known, the unknowns for our current study are the potential temperature θ, the concentrations of the water vapor, cloud-condensate and rain water q_v, q_c, q_r and the saturation concentration q_{vs}. If T is the temperature then we classically have

$$\theta = T(\frac{p_0}{p})^\kappa = \frac{T}{\Pi}, \ \Pi = (\frac{p}{p_0})^\kappa, \tag{2.1}$$

where $\kappa = (\gamma - 1)/\gamma$ and $\gamma = c_p/c_v$ is the ratio of specific heats at constant pressure and at constant volume.

Before going any further, we shall first make some simple observations. Of course, the quantities q_v, q_c, q_r, q_{vs} being relative mass fractions ratios take their values in the interval $[0, 1]$. Furthermore, the air can not be supersaturated (in general). In other words, we have the constraint $0 \leqslant q_v \leqslant q_{vs}$.

Following e.g. [MP74] or [KW78] (see in particular (2.5) in [KW78]), the general form of the equations for q_v, q_c, q_r is given by

$$\frac{dq}{dt} = M_q + D_q. \tag{2.2}$$

In (2.2), the symbol $\frac{d}{dt}$ is the material derivative given here by $\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}$, i.e.,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}, \tag{2.3}$$

where $\omega = \frac{dp}{dt}$. Corresponding to q_v, q_c and q_r, the terms D_q are the usual dissipation terms (like the $3D$ Laplacian Δ_3) and the quantities M_q are the rates of the production of species q, which are described below using the notations in [KW78] (see (2.9b) – (2.9d)):

$$\begin{cases} M_{q_v} = \delta\frac{dq_{vs}}{dt} + E_r, \\ M_{q_c} = -\delta\frac{dq_{vs}}{dt} - A_r - C_r, \\ M_{q_r} = -g\frac{\partial}{\partial p}(\rho q_r V_t) - E_r + A_r + C_r \,. \end{cases} \tag{2.4}$$

Note that in (2.4), ρ is given by the ideal gas law:

$$p = \rho R T, \quad \rho = \frac{p}{RT}, \tag{2.5}$$

and that T in (2.5) is the total temperature and not the deviation T' appearing in (2.12) below; V_t appearing in the expression of M_{q_r} is the terminal velocity of the falling rain. The term A_r is the rate of auto-conversion of the rain water; C_r is the rate of the collection of cloud water by falling rain; and E_r represents the evaporation rate of the rain water. Most important for our study is the term $\delta\frac{dq_{vs}}{dt}$, which represents the rate of condensation or evaporation of the cloud water, and this occurs only when $\omega < 0$ falling water and $q_v \geqslant q_{vs}$. Hence the coefficient δ is defined as:

$$\begin{aligned} &\delta = 1, \ \text{ if } \ \omega < 0 \ \text{ and } \ q_v \geqslant q_{vs}\,, \\ &\delta = 0, \ \text{ if } \ \omega \geqslant 0 \ \text{ or } \ q_v < q_{vs}\,. \end{aligned} \tag{2.6}$$

The function q_{vs} is a diagnostic variable; it is explicitly given at each instant of time as a function of p and T (or θ), that is

$$q_{vs} = Q_{vs}(p, T). \tag{2.7}$$

The expression of q_{vs} as a function of T and p results from the application of the Clausius–Clapeyron equation. According to ([RY89], p. 14), q_{vs} can be expressed as a function of the saturation vapor pressure e_{vs}

$$q_{vs} = \frac{3.8}{p - 0.378\, e_{vs}} \frac{e_{vs}}{6.11} = 0.6219 \frac{e_{vs}}{p - 0.378\, e_{vs}}, \tag{2.8}$$

where by Tetens' formula (see (2.11) of [KW78]),

$$e_{vs} = 6.11 \exp\Big(a \frac{T - 273}{T - b}\Big). \tag{2.9}$$

Here T is in Kelvin, $a = 17.27$, $b = 35.5$. Because we only consider the above freezing case in our model, $b \ll 273$, $b \ll T$, and from now on we set $b = 0$ for simplicity.

Remark 2.1. We see that e_s is a strictly positive, bounded and smooth function of the temperature T for the temperature ranges found in the troposphere. Considering the usual range for pressure p, e.g. $200 \leqslant p \leqslant 1000$ (see [BCHTT15]), we can avoid the possible singularity at $p = 0.378\, e_{vs}$ in (2.8) by a suitable modification of (2.9) outside the physical relevant values of T (see $\varphi(T)$ in Remark 2.3). So q_{vs} is a positive smooth and increasing function of e_{vs}, which in turn implies that $q_{vs} = Q_{vs}(p, T)$ is a positive bounded smooth function of p and T for all values of $p \geqslant 0$ and $T \in \mathbb{R}$. In particular, we will use the properties that $Q_{vs}(p, T)$ has bounded first order and second order partial derivatives with respect to the variables p and T in Section 4.

Now we need to find the expression dq_{vs}/dt which appears in the right-hand sides of equations (2.4) and (2.11). As in [Hal71] and [HW80], the expression of $\frac{dq_{vs}}{dt}$ results from combining the first law of thermodynamics with the Clausius-Clapeyron equations, so that

$$\frac{dq_{vs}}{dt} = F(p, T) = \frac{q_{vs} T}{p} \Big(\frac{LR - c_p R_v T}{c_p R_v T^2 + q_{vs} L^2} \Big) \omega, \tag{2.10}$$

where p and T are the pressure and temperature; L is the latent heat of vaporization; R, R_v are the gas constants for dry air and water vapor respectively and c_p represents the specific heat of dry air at constant pressure.

Finally we have to supplement the equations above with the equation for the temperature T. In fact, instead of T, we consider the potential temperature θ previously introduced, or more precisely the deviation $\theta^{'} = \theta - \theta_h$, where θ_h is a reference temperature.

Hence, as with (2.2) and (2.4), the equation for θ' is given by

$$\frac{d\theta'}{dt} = -\frac{\Pi R}{g^2\Phi}\omega N_h^2\theta_h + M_\theta + D_\theta + \frac{\dot{Q}}{c_p\Pi}, \tag{2.11}$$

with

$$M_\theta = -\frac{L}{c_p\Pi}(\delta\frac{dq_{vs}}{dt} + E_r);$$

(see e.g. (2.9a) in [KW78]). Also, following [KW78] (see (2.2) – (2.3) in [KW78]) we have

$$\theta' = \frac{T'}{\Pi}, \quad \theta_h = \frac{T_h}{\Pi}, \tag{2.12}$$

where $T = T_h + T'$; see also equation (6) in [MP74]. Furthermore we have (see [Xue89], (1.2.27)-(1.2.28)):

$$N_h^2 = -\frac{gr_h}{\theta_h}\frac{\partial\theta_h}{\partial p}, \quad r_h = g\rho_h = \frac{gp}{RT_h}. \tag{2.13}$$

The source terms. The coefficients describing the microphysics A_r, C_r, E_r and the terminal velocity V_t are defined empirically. Common expressions of these terms are as follows (see e.g. [KW78]):

$$A_r = k_1(q_c - q_{crit})^+, \quad C_r = k_2 q_c q_r^{0.875}, \quad E_r = k_3(q_r^+)^{0.5}(q_{vs} - q_v)^+, \tag{2.14}$$

$$V_t = 5.32\, q_r^{0.2}. \tag{2.15}$$

We observe that all these quantities are continuous functions of $U = (q_v, q_c, q_r, \theta^{'})$. We will slightly modify some terms in a way which simplifies the mathematical study but does not modify the physical relevance of the equations. For example, after a suitable extension outside the physical relevant values of q_v, q_c, q_r, $\theta^{'}$, all what we need is to assume that the coefficients are continuous bounded functions of U, compactly supported in the region of $\mathbb{R}^3$ corresponding to q_v, q_c, q_r.

Remark 2.2. For mathematical convenience and in agreement with the physical meaning of q_r $(0 \leqslant q_r \leqslant 1)$, we will replace q_r in (2.14) by $\tau(q_r) = 0$ if $q_r \leqslant 0$; $= q_r$ if $0 \leqslant q_r \leqslant 1$; and $= 1$ if $q_r \geqslant 1$.

Remark 2.3. We similarly need to comment on (2.10) and change its expression outside the physically relevant values of q_{vs}, T and ω. Firstly, since (2.10) is only relevant for

$\omega < 0$, we replace ω by $-\omega^-$. Then to avoid a possible singularity at $T = 0$, we replace T by $\varphi(T)$, where φ is a smooth (e.g. C^2) positive real function with $\varphi(T)$:

$$\begin{cases} = T & for\ T_* \leqslant T \leqslant T_{**}, \\ \geqslant T_*/2 & for\ T \leqslant T_*, \\ = 0 & for\ T \geqslant 2T_{**}. \end{cases} \tag{2.16}$$

Here $T_* > 0$ is smaller than any temperature on earth (e.g. 100K) and T_{**} is larger than any temperature on earth (e.g. 355K). We see that the function $F(p, T)$ in (2.10) is actually uniformly bounded in p and T. Moreover, replacing T by $\varphi(T)$ in (2.10) and for a given initial value $q_{vs}(0) \geqslant 0$ we see that the modified equation (2.10) gives q_{vs} as a positive smooth (C^2) bounded function of T and p after we integrate (2.10):

$$q_{vs}(p, T) = Q_{vs}(p, T) \geqslant 0. \tag{2.17}$$

Remark 2.4. Some authors consider, instead of the expression of E_r in (2.14),

$$E_r = k_3 T (q_r^+)^\beta (q_{vs} - q_v)^+, \ \ \beta \in (0, 1], \tag{2.18}$$

see e.g. [HKLT17]. We could likewise consider this form of E_r if we replace T by $\varphi(T)$ which is physically equivalent as we already discussed.

The rest of the article is organized as follows. In Sections 3 and 4 we develop the mathematical setting for these equations. Sections 3 is devoted to presenting the general mathematical setting, the initial and boundary conditions, and the handling of the quantity δ by using Heaviside functions, in continuation of [CT12] and [CFTT13]. In Section 4, we account for the constraint $q_v \leqslant q_{vs}$ and introduce the quasi-variational inequality that we intend to study, that is, prove the existence of its solution. To this aim, we introduce, in Section 4, a penalization procedure, by which we approximate the quasi variational inequality by a relatively standard nonlinear problem which can be treated by classical methods. Note that the use of the penalization method is a convenient mathematical tool and we do not try to give a physical meaning to the penalized problem. Penalization has been introduced by R. Courant [Cou43] and it is very common in Optimization Theory (see e.g. [Cea78], [Kar11] and [PT80]). Then we prove some a priori estimates for the penalized ($\varepsilon-$regularized) solution, and finally pass to the limit as $\varepsilon \to 0$ to end up with the existence of the solution for the initial (non regularized) problem. The passage to the limit relies on using some classical compactness results and convex analysis tools. Other properties concerning the solutions such as uniqueness, maximum principle, etc., will be addressed elsewhere.

3. Discontinuity and boundary value problem

In this part and the next one, we will consider the above equations which will be supplemented with initial and boundary conditions. There are two additional issues:

(1) The coefficient δ which is discontinuous and that we will replace by a Heaviside function, as in [CT12], [CFTT13].
(2) The inequality constraint on the variable q_v: $q_v \leqslant q_{vs}$.

The first issue is addressed in this section, the second issue is addressed in Section 4.

We are going to write the differential equations and the boundary value problems for the quantities q_v, q_c, q_r, θ'. We assume for simplicity that the velocity $\mathbf{u} = (\mathbf{v}, \omega)$ is prescribed, otherwise we would have to add the equations governing the evolution of $\mathbf{u}$.

The flow takes place in a domain $\mathcal{M}$ in the (x, y, p)-space, $\mathcal{M} = \mathcal{M}' \times (p_0, p_1)$, where $\mathcal{M}' \subset \mathbb{R}^2$ is smooth and bounded, and $0 < p_0 \leqslant p \leqslant p_1$ is the range of values of p that we consider; here p_0, p_1 are two fixed real numbers. We use "$\mathbf{n}$" to denote the outward normal vector field to the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ which consists of three parts $\Gamma_u, \Gamma_i, \Gamma_l$, namely the upper and lower interface with the ocean and the lateral components of the boundary. They are defined by

$$\begin{aligned} \Gamma_u &= \{(x,y,p) \in \overline{\mathcal{M}}; p = p_0\}, \\ \Gamma_i &= \{(x,y,p) \in \overline{\mathcal{M}}; p = p_1\}, \\ \Gamma_l &= \{(x,y,p) \in \overline{\mathcal{M}}; p_0 \leqslant p \leqslant p_1, (x,y) \in \partial\mathcal{M}'\}. \end{aligned} \tag{3.1}$$

We set $\nabla = (\partial_x, \partial_y)$ and $\Delta = \partial_x^2 + \partial_y^2$ to be the horizontal gradient and horizontal Laplace operators, respectively and $\nabla_3 = (\nabla, \partial_p)$, $\Delta_3 = \Delta + \partial_p^2$ to be the 3D gradient and Laplace operators, respectively. In this way, the heat and vapor diffusion operators $\mathcal{A}_\theta$ and $\mathcal{A}_q$ are described as

$$\mathcal{A}_\theta = -\mu_\theta \Delta - \nu_\theta \partial_p\big((\frac{gp}{R\bar\theta})^2 \partial_p\big),\ \mathcal{A}_q = -\mu_q \Delta - \nu_q \partial_p\big((\frac{gp}{R\bar\theta})^2 \partial_p\big), \tag{3.2}$$

where μ_q, ν_q $(q \in \{q_v, q_c, q_r\})$, $\mu_\theta, \nu_\theta, g, R, c_p$ are all positive constants and $\bar\theta = \bar\theta(p)$ is the average potential temperature over the isobar with pressure p. We assume that $\bar\theta$ satisfies:

$$\bar\theta_* \leqslant \bar\theta(p) \leqslant \bar\theta^*,\ |\partial_p \bar\theta(p)| \leqslant M, \text{ for some positive constants } \bar\theta_*, \bar\theta^*, M \text{ and } p \in [p_0, p_1]. \tag{3.3}$$

We set $U = (q_v, q_c, q_r, \theta')$. We will now describe in details the boundary value problem for each of the quantities under consideration.

3.1. **The equation for q_v.** The equation for q_v is written

$$\begin{aligned} \frac{\partial q_v}{\partial t} + \mathcal{A}_{q_v} q_v + \mathbf{v}\cdot\nabla q_v + \omega \frac{\partial q_v}{\partial p} &\in f_{q_v}(q_v, q_c, q_r, \theta') + F\mathcal{H}(q_v - q_{vs}) \\ &= f_{q_v}(U) + F\mathcal{H}(q_v - q_{vs}), \end{aligned} \tag{3.4}$$

where $\mathcal{H}$ is the multi-valued Heaviside function such that $\mathcal{H} = [0,1]$ at 0 and (see (2.10) as well as Remark 2.3):

$$F = F(T,p) = -\omega^- \frac{q_{vs}\varphi(T)}{p}\big(\frac{LR - c_p R_v \varphi(T)}{c_p R_v \varphi(T)^2 + q_{vs} L^2}\big), \tag{3.5}$$

$$f_{q_v}(U) = f_{q_v}(q_v, q_c, q_r, \theta') = E_r. \tag{3.6}$$

We will use the form $E_r = k_3\tau(q_r)^{0.5}(q_{vs} - q_v)^+$ as indicated in (2.14). Note that we have replaced $(q_r)^+$ by $\tau(q_r)$ according to Remark 2.2.

We consider the following boundary conditions to be associated with the above equation:

$$\partial_p q_v = \beta_v(q_{v*} - q_v) \text{ on } \Gamma_i, \quad \partial_p q_v = 0 \text{ on } \Gamma_u, \quad \partial_{n_v} q_v = 0 \text{ on } \Gamma_l, \tag{3.7}$$

where $\partial_{n_v} = \partial_{n_{\mathcal{A}_{q_v}}}$ is the co-normal derivative associated with $\mathcal{A}_{q_v}$ which reduces on Γ_l to

$$-\mu_{q_v}\mathbf{n}_H \cdot \nabla q_v, \tag{3.8}$$

where $\mathbf{n}_H$ is the horizontal component of the unit outward normal $\mathbf{n}$ on $\mathcal{M}$ (that is the unit outward normal on Γ_l).

We also associate with (3.4) the following initial condition

$$q_v(x, y, p, 0) = q_{v0}(x, y, p). \tag{3.9}$$

In (3.7), $q_{v*} = q_{v*}(x, y, t)$ is a specific humidity distribution at the bottom of the atmosphere and β_v is a given positive constant.

3.2. **The equation for q_c.** The equation for q_c is written

$$\begin{aligned}\frac{\partial q_c}{\partial t} + \mathcal{A}_{q_c} q_c + \mathbf{v} \cdot \nabla q_c + \omega \frac{\partial q_c}{\partial p} &\in f_{q_c}(q_v, q_c, q_r, \theta') - F\mathcal{H}(q_v - q_{vs}) \\ &= f_{q_c}(U) - F\mathcal{H}(q_v - q_{vs}),\end{aligned} \tag{3.10}$$

where F, $\mathcal{H}$ are defined below (3.4) and

$$f_{q_c}(U) = f_{q_c}(q_v, q_c, q_r, \theta') = -k_1(q_c - q_{crit})^+ - k_2 q_c \tau(q_r)^{0.875}. \tag{3.11}$$

Similar to what we did for E_r, here we have replaced q_r in C_r by $\tau(q_r)$ (compare to (2.14)).

We supplement the above equation with the following natural boundary conditions

$$\partial_p q_c = \beta_c(q_{c*} - q_c) \text{ on } \Gamma_i, \quad \partial_p q_c = 0 \text{ on } \Gamma_u, \quad \partial_{n_c} q_c = 0 \text{ on } \Gamma_l, \tag{3.12}$$

and the initial condition

$$q_c(x, y, p, 0) = q_{c0}(x, y, p). \tag{3.13}$$

In (3.12), $q_{c*} = q_{c*}(x, y, t)$ is a critical specific humidity distribution at the bottom of the atmosphere and β_c is a given positive constant, and $\partial_{n_c} q_c$ is defined as $\partial_{n_v} q_v$ in (3.8).

3.3. **The equation for q_r.** The equation for q_r is written

$$\frac{\partial q_r}{\partial t} + \mathcal{A}_{q_r} q_r + \mathbf{v} \cdot \nabla q_r + \omega \frac{\partial q_r}{\partial p} = -g\frac{\partial}{\partial p}(\rho q_r V_t) - E_r + A_r + C_r. \tag{3.14}$$

Here, we will continue to use the expression $E_r = k_3\tau(q_r)^{0.5}(q_{vs} - q_v)^+$ in accordance with the q_v-equation.

By (2.15), we have

$$\frac{1}{5.32}\frac{\partial}{\partial p}(\rho q_r V_t) = \frac{\partial}{\partial p}\big(\frac{p\,q_r^{1.2}}{R\Pi\theta}\big) = \frac{q_r^{1.2}}{R\Pi\theta} + \frac{1.2p\,q_r^{0.2}}{\Pi\theta R}\frac{\partial q_r}{\partial p} - \frac{p\,q_r^{1.2}}{R\Pi\theta^2}\big(\theta\frac{\kappa}{p} + \frac{\partial\theta_h(p)}{\partial p} + \frac{\partial\theta'}{\partial p}\big). \quad (3.15)$$

Referring to (3.15) and replacing again q_r by $\tau(q_r)$, the equation for q_r takes the following form

$$\frac{\partial q_r}{\partial t} + \mathcal{A}_{q_r}q_r + \mathbf{v}\cdot\nabla q_r + \omega\frac{\partial q_r}{\partial p} = -5.32\,g\frac{\partial}{\partial p}(\rho q_r^{1.2}) - k_3\tau(q_r)^{0.5}(q_{vs}-q_v)^+ + k_1(q_c - q_{crit})^+ + k_2 q_c\tau(q_r)^{0.875}. \quad (3.16)$$

By the ideal gas law (2.5), the above equation can be further transformed to

$$\frac{\partial q_r}{\partial t} + \mathcal{A}_{q_r}q_r + \mathbf{v}\cdot\nabla q_r + \omega\frac{\partial q_r}{\partial p} = -5.32\,g\frac{\partial}{\partial p}\big(\frac{pq_r^{1.2}}{R\Pi\theta}\big) - k_3\tau(q_r)^{0.5}(q_{vs}-q_v)^+ + k_1(q_c - q_{crit})^+ + k_2 q_c\tau(q_r)^{0.875}. \quad (3.17)$$

By (3.15), we obtain

$$\frac{\partial q_r}{\partial t} + \mathcal{A}_{q_r}q_r + \mathbf{v}\cdot\nabla q_r + \omega\frac{\partial q_r}{\partial p} = f_{q_r}(q_v, q_c, q_r, \theta') = f_{q_r}(U), \quad (3.18)$$

where

$$\begin{aligned} f_{q_r}(U) &= f_{q_r}(q_v, q_c, q_r, \theta') \\ &= -5.32\,g\big(\frac{\tau(q_r)^{1.2}}{R\Pi\theta_\alpha} + \frac{1.2p\,\tau(q_r)^{0.2}}{R\Pi\theta_\alpha}\frac{\partial q_r}{\partial p} - \frac{p\,\tau(q_r)^{1.2}}{R\Pi\theta_\alpha^2}\big(\theta_\alpha\frac{\kappa}{p} + \frac{\partial\theta_h(p)}{\partial p} + \frac{\partial\theta'}{\partial p}\big)\big) \\ &\quad - k_3\tau(q_r)^{0.5}(q_{vs}-q_v)^+ + k_1(q_c - q_{crit})^+ + k_2 q_c\tau(q_r)^{0.875}. \end{aligned} \quad (3.19)$$

In (3.19) we have also replaced θ by $\theta_{\wedge\alpha} = min(\theta, \alpha)$ where $\alpha > 0$ is less than any temperature on earth. This is physically relevant and mathematically useful.

We supplement equation (3.18) with the following boundary conditions and initial conditions:

$$\partial_p q_r = \beta_r(q_{r*} - q_r) \text{ on } \Gamma_i, \quad \partial_p q_r = 0 \text{ on } \Gamma_u, \quad \partial_{n_r} q_r = 0 \text{ on } \Gamma_l, \quad (3.20)$$

$$q_r(x, y, p, 0) = q_{r0}(x, y, p). \quad (3.21)$$

Here $q_{r*} = q_{r*}(x, y, t)$ is a specific humidity distribution at the bottom of the atmosphere; β_r is a given positive constant. Also $\partial_{n_r} q_r$ is defined as $\partial_{n_v} q_v$ in (3.8).

3.4. **The equation for θ ($\sim \theta'$).** The deviation θ' from the reference state $\theta_h(p)$ satisfies the following equation

$$\begin{aligned} \frac{\partial\theta'}{\partial t} + \mathcal{A}_\theta\theta' + \mathbf{v}\cdot\nabla\theta' + \omega\frac{\partial\theta'}{\partial p} &\in f_{\theta'}(q_v, q_c, q_r, \theta') - \frac{L}{c_p\Pi}F\mathcal{H}(q_v - q_{vs}) \\ &= f_{\theta'}(U) - \frac{L}{c_p\Pi}F\mathcal{H}(q_v - q_{vs}), \end{aligned} \quad (3.22)$$

where

$$f_{\theta'}(U) = f_{\theta'}(q_v, q_c, q_r, \theta') = -\frac{\theta_h N_h^2}{g}\omega - \frac{L}{c_p \Pi}\left(k_3 \tau (q_r)^{0.5} (q_{vs} - q_v)^+\right) + f_\theta. \tag{3.23}$$

Here $f_\theta = f_\theta^1 + f_\theta^2$, with f_θ^1 a source term and $f_\theta^2 = \omega \frac{\partial \theta_h(p)}{\partial p}$ after observing that $\mathcal{A}_\theta \theta_h(p) + \mathbf{v} \cdot \nabla \theta_h(p)$ vanishes.

We consider the boundary conditions

$$\partial_p \theta' = \alpha(\theta'_* - \theta') \text{ on } \Gamma_i, \quad \partial_p \theta' = 0 \text{ on } \Gamma_u, \quad \partial_{n_{\theta'}} \theta' = 0 \text{ on } \Gamma_l, \tag{3.24}$$

and initial condition

$$\theta'(x, y, p, 0) = \theta'_0(x, y, p). \tag{3.25}$$

Here the function $\theta'_* = \theta'_*(x, y, t)$ is a typical potential temperature; α is a given positive constant, and $\partial_{n_{\theta'}} \theta'$ is defined as $\partial_{n_v} q_v$ in (3.8).

4. Variational and weak formulation of the problem.

From the mathematical point of view, a new difficulty as compared to [CT12], [CFTT13] is the constraint $q_v \leqslant q_{vs}$ which leads us to the concept of *quasi variational inequality* (instead of a variational inequality). Indeed in the notations below the velocity $\mathbf{u}$ is still given and the set of unknowns U consists of q_v, q_r, q_c and the temperature T ; in fact we rather consider the potential temperature θ, and replace it by the difference $\theta' = \theta - \theta_h$ between θ and a reference temperature θ_h. Hence $U = (q_v, q_r, q_c, \theta')$. Now, as recalled in Remark 2.3, the saturation concentration q_{vs} is itself a function of T and p, which we express as $q_{vs} = Q_{vs}(p, U)$ or $q_{vs} = Q_{vs}(p, T)$ for simplicity. Hence the constraint $q_v \leqslant q_{vs}$ appears as a quasi variational inequality where the solution U is subject to belonging to a convex set which depends itself on the solution:

$$U \in \mathcal{K} = \mathcal{K}(U).$$

Quasi variational inequalities have been introduced by Bensoussan and Lions, motivated by the study of economical problems [BL76], [BL77]; see also [BF78], [BL84], [BL73a], [BL73b], [BL74] and [BL75]. Subsequently quasi variational inequalities have been used for problems in mechanics, physics and imagery, see e.g. [KN07], [Kan14], [Mil14] and [LLBS14].

We start in Section 4.1 by giving the weak form of the problem and then in Section 4.2 we account for the constraint $U \in \mathcal{K}(U)$ and introduce the quasi-variational inequality.

4.1. **Notations.** We denote as usual $H = L^2(\mathcal{M})$, $V = H^1(\mathcal{M})$ and we set $\mathbb{H} = H \times H \times H \times H$ and $\mathbb{V} = V \times V \times V \times V$. We use $(\cdot,\cdot)_{L^2}$ (regarded the same as $(\cdot,\cdot)_H$) and $|\cdot|_{L^2}$ to denote the usual scalar product and induced norm in H. In the space V, we will use $((\cdot,\cdot))$ to denote the scalar product adapted to the problem under investigation

$$((\varphi, \phi)) := (\nabla\varphi, \nabla\phi) + (\partial_p\varphi, \partial_p\phi) + \int_{\Gamma_i} \varphi\phi \, d\Gamma_i,$$

and the induced norm is denoted $\|\cdot\|$. The symbol $\langle\cdot,\cdot\rangle$ will denote the duality pair between a Banach space E and its dual space E^*. Associated with the Navier-Stokes equations, we also use the following standard notations:

$$\mathbf{H} = \{\mathbf{u} \in H \times H \times H \,\big|\, div\,\mathbf{u} = 0 \text{ and } \mathbf{u}\cdot n = 0 \text{ on } \partial\mathcal{M}\},$$

$$\mathbf{V} = \{\mathbf{u} \in V \times V \times V \,\big|\, div\,\mathbf{u} = \mathbf{0} \text{ and } \mathbf{u}\cdot \mathbf{n} = \mathbf{0} \text{ on } \partial\mathcal{M}\},$$

which will serve as the natural function spaces for the vector field $\mathbf{u}$. In fact we will assume that

$$\mathbf{u} \in L^\infty(0,t_1;H^1(\mathcal{M})^3) \cap L^\infty((0,t_1)\times\mathcal{M}). \tag{4.1}$$

In view of deriving the weak (variational) formulation of the boundary value problem, we multiply e.g. the expression $\mathcal{A}_{q_v}q_v$ by a test function q_v^b. Assuming smoothness and taking into account the boundary conditions (3.7) for q_v we find

$$\begin{aligned}\langle\mathcal{A}_{q_v}q_v, q_v^b\rangle &= \Big(-\mu_{q_v}\Delta - \nu_{q_v}\partial_p\big((\frac{gp}{R\bar{\theta}})^2\partial_p\big), q_v^b\Big)\\ &:= \mu_{q_v}(\nabla q_v, \nabla q_v^b)_H + \nu_{q_v}\int_{\mathcal{M}}\big(\frac{gp}{R\bar{\theta}}\big)^2\partial_p q_v\partial_p q_v^b\,d\mathcal{M}\\ &\quad + \nu_{q_v}\int_{\Gamma_i}\big(\frac{gp_1}{R\bar{\theta}}\big)^2\beta_{q_v}(q_v - q_{v*})q_v^b\,d\Gamma_i.\end{aligned} \tag{4.2}$$

We do the same for q_c, q_r and θ' and thus

$$\langle\mathcal{A}_{q_c}q_c, q_c^b\rangle = \mu_{q_c}(\nabla q_c, \nabla q_c^b)_H + \nu_{q_c}\int_{\mathcal{M}}\big(\frac{gp}{R\bar{\theta}}\big)^2\partial_p q_c\partial_p q_c^b\,d\mathcal{M} + \nu_{q_c}\int_{\Gamma_i}\big(\frac{gp_1}{R\bar{\theta}}\big)^2\beta_{q_c}(q_c - q_{c*})q_c^b\,d\Gamma_i, \tag{4.3}$$

$$\langle\mathcal{A}_{q_r}q_r, q_r^b\rangle = \mu_{q_r}(\nabla q_r, \nabla q_r^b)_H + \nu_{q_r}\int_{\mathcal{M}}\big(\frac{gp}{R\bar{\theta}}\big)^2\partial_p q_r\partial_p q_r^b\,d\mathcal{M} + \nu_{q_r}\int_{\Gamma_i}\big(\frac{gp_1}{R\bar{\theta}}\big)^2\beta_{q_r}(q_r - q_{r*})q_r^b\,d\Gamma_i, \tag{4.4}$$

and

$$\langle\mathcal{A}_{\theta}\theta', \theta'^b\rangle = \mu_{\theta}(\nabla \theta', \nabla \theta'^b)_H + \nu_{\theta}\int_{\mathcal{M}}\big(\frac{gp}{R\bar{\theta}}\big)^2\partial_p \theta'\partial_p \theta'^b\,d\mathcal{M} + \nu_{\theta}\int_{\Gamma_i}\big(\frac{gp_1}{R\bar{\theta}}\big)^2\alpha(\theta' - \theta'_*)\theta'^b\,d\Gamma_i. \tag{4.5}$$

Consequently, we define the following bilinear forms

$$a_\theta(\theta', \theta'^b) = \mu_{\theta}(\nabla \theta', \nabla \theta'^b)_H + \nu_{\theta}\int_{\mathcal{M}}\big(\frac{gp}{R\bar{\theta}}\big)^2\partial_p \theta'\partial_p \theta'^b\,d\mathcal{M} + \nu_\theta\alpha\int_{\Gamma_i}\big(\frac{gp_1}{R\bar{\theta}}\big)^2\theta'\theta'^b\,d\Gamma_i, \tag{4.6}$$

$$a_q(q, q^b) = \mu_{q}(\nabla q, \nabla q^b)_H + \nu_{q}\int_{\mathcal{M}}\big(\frac{gp}{R\bar{\theta}}\big)^2\partial_p q\partial_p q^b\,d\mathcal{M} + \nu_q\beta_q\int_{\Gamma_i}\big(\frac{gp_1}{R\bar{\theta}}\big)^2 qq^b\,d\Gamma_i. \tag{4.7}$$

Similarly, we define $b(\mathbf{u}, \psi, \psi^b)$ as follows:

$$b(\mathbf{u}, \psi, \psi^b) = \int_{\mathcal{M}} (\mathbf{v} \cdot \nabla \psi + \omega \partial_p \psi) \psi^b \, d\mathcal{M}, \tag{4.8}$$

which we will use with $(\psi, \psi^b) = (\theta', \theta'^b), (q_v, q_v^b), (q_r, q_r^b), (q_c, q_c^b)$. We recall here that $\mathbf{u} = (\mathbf{v}, \omega)$ is the three dimensional velocity, $\mathbf{v}$ is the horizontal velocity and ω is the vertical velocity of the air in the x, y, p system.

Analogously, we define the linear functionals:

$$l_\theta(\theta'^b) = \nu_\theta \alpha \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 \theta_* \theta'^b \, d\Gamma_i, \quad l_q(q^b) = \nu_q \beta_q \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 q_* q^b \, d\Gamma_i, \tag{4.9}$$

$$l(U^b) = l_{q_c}(q_c^b) + l_{q_v}(q_v^b) + l_{q_r}(q_r^b) + l_\theta(\theta'^b), \tag{4.10}$$

which correspond to the constant terms in $\mathcal{A}_\theta$, $\mathcal{A}_q$ and $\mathcal{A}$ respectively.

We introduce the multilinear forms for U and $U^b = (q_c^b, q_v^b, q_r^b, \theta'^b)$

$$a(U, U^b) = a_{q_c}(q_c, q_c^b) + a_{q_v}(q_v, q_c^b) + a_{q_r}(q_r, q_r^b) + a_\theta(\theta', \theta'^b), \tag{4.11}$$

$$b(\mathbf{u}, U, U^b) = \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_{x,y,p} U) \cdot U^b \, d\mathcal{M}. \tag{4.12}$$

It is easy to see that

$$b(\mathbf{u}, U, U^b) = b(\mathbf{u}, q_c, q_c^b) + b(\mathbf{u}, q_v, q_v^b) + b(\mathbf{u}, q_r, q_r^b) + b(\mathbf{u}, \theta', \theta'^b). \tag{4.13}$$

In view of $\nabla \cdot \mathbf{u} = 0$, we readily see by performing integration by parts that

$$b(\mathbf{u}, \psi, \psi) = 0, \ \forall \ \psi \in V. \tag{4.14}$$

Before we move further, we first give the following well-known estimates.

More precisely, we have the following lemma concerning the boundedness of the above functionals.

Lemma 4.1. *Assume $U = (q_v, q_c, q_r, \theta'), U^b = (q_v^b, q_c^b, q_r^b, \theta'^b) \in \mathbb{V}$ and $\mathbf{u} \in \mathbf{V}$. There exist universal positive constants λ and κ such that (q denotes here q_v, q_c or q_r):*

$$|a_\theta(\theta, \theta^b)| \leqslant \kappa \|\theta'\| \|\theta^b\|, \ a_\theta(\theta, \theta) \geqslant \lambda \|\theta\|^2; \tag{4.15}$$

$$|a_q(q, q^b)| \leqslant \kappa \|q\| \|q^b\|, \ a_q(q, q) \geqslant \lambda \|q\|^2; \tag{4.16}$$

$$|b(\mathbf{u}, U, U^b)| \leqslant \kappa \|\mathbf{u}\|_{\mathbf{V}} |U|_{L^2}^{\frac{1}{2}} \|U\|^{\frac{1}{2}} \|U'^b\|; \tag{4.17}$$

$$|l_\theta(\theta'^b)| \leqslant \kappa \|\theta'^b\|, \ |l_q(q^b)| \leqslant \kappa \|q^b\|. \tag{4.18}$$

The proof of Lemma 4.1 is based on a routine use of the Cauchy-Schwarz inequality and the trace theorem. We shall omit the details here.

It is well-known that the linear operators A_θ, $A_q : V \to V^*$ defined through the relations

$$\langle A_\theta u, v \rangle := a_\theta(u, v), \ \langle A_q u, v \rangle := a_q(u, v), \forall \, u, v \in V, \tag{4.19}$$

are both bounded linear operators.

Similarly, the operator $B(\mathbf{u}, U) = \big(b(\mathbf{u}, U), b(\mathbf{u}, q)\big) : V \times \mathbb{V} \to \mathbb{V}^*$ defined by

$$\langle B(\mathbf{u}, U), U^b \rangle := \big(b(\mathbf{u}, \theta', \theta'^b), b(\mathbf{u}, q, q^b)\big) \; \forall \, \mathbf{u} \in V, U, U^b \in \mathbb{V}, \tag{4.20}$$

where $\mathbb{V}^*$ is the dual space of $\mathbb{V}$.

4.2. **Weak formulation of the problem.** Our equations can be written in the following compact form

$$\partial_t U + \mathcal{A} U + \mathbf{u} \cdot \nabla_{\mathbf{x}} \cdot U \in f(U, \theta') + \mathcal{F}\mathcal{H}(q_v - q_{vs}), \tag{4.21}$$

where $\mathcal{F}$ is the vector $(F, -F, 0, -\frac{L}{c_p \Pi} F)^t$.

Alternatively, (4.21) means that there exists single-valued Heaviside function $h_{q_v} \in \mathcal{H}(q_v - q_{vs})$ taking values in $[0, 1]$ such that

$$\partial_t U + \mathcal{A} U + \mathbf{u} \cdot \nabla_{\mathbf{x}} \cdot U = f(U, \theta') + \mathcal{F} h_{q_v}. \tag{4.22}$$

If we adopt the following notations for $U_0 = U_0(x, y, p)$, $U_* = U_*(x, y, p)$ and $U = U(x, y, p)$

$$U_0 = (q_{v0}, q_{c0}, q_{r0}, \theta'_0)^t, \; U_* = (q_{v*}, q_{c*}, q_{r*}, \theta'_*)^t,$$

and define the coefficient matrix $\mathcal{C} = \text{diag}\{\beta_c, \beta_v, \beta_r, \alpha\}$, then the initial and boundary conditions associated with the system (4.21) can be written as follows

$$U(x, y, p, 0) = U_0(x, y, p), \tag{4.23}$$

$$\partial_p U = \mathcal{C}(U_* - U) \text{ on } \Gamma_i, \quad \partial_{n_\mathcal{A}} U = 0 \text{ on } \Gamma_u \cup \Gamma_l, \tag{4.24}$$

where $\partial_{n_\mathcal{A}} U$ is defined componentwise as in (3.8).

For the weak formulation we will treat differently the equations for $\bar{U} = (q_c, q_r, \theta')$ and the equation for q_v which is subjected to the constraint $q_v \leqslant q_s$.

For $\bar{U}$, we consider the equations (3.10), (3.18), (3.22) for q_c, q_r, θ', respectively, and multiply them by test functions q_c^b, q_r^b, θ'^b. Assuming smoothness as before, we obtain in view of (4.3)–(4.5),

$$\int_0^{t_1} \big[\langle \partial_t \bar{U}, \bar{U}^b \rangle + \bar{a}(\bar{U}, \bar{U}^b) + \bar{b}(\mathbf{u}, \bar{U}, \bar{U}^b) - \bar{l}(\bar{U}^b)\big] \, dt = \int_0^{t_1} (f(\bar{U}) + \bar{\mathcal{F}} h_{q_v}, \bar{U}^b) dt, \tag{4.25}$$

for all $\bar{U}^b \in L^2(0, t_1; (H^1)^3)$ and

$$\bar{U}(t = 0) = \bar{U}_0. \tag{4.26}$$

Recall again that here $\bar{l}$ represents the constant part of the operator $\bar{\mathcal{A}}$ and $\bar{\mathcal{F}}$ represents the vector $(-F, 0, -\frac{L}{c_p \Pi} F)^t$.

With the constraint $q_v \leqslant q_{vs}$, and by analogy with what was done in [TT16] when q_{vs} is constant and $0 \leqslant q_v \leqslant q_{vs}$, we can weaken (3.4) in the form:

$$\mathcal{L}(q_v) \leqslant f_{q_v}(U) + F h_{q_v}, \tag{4.27}$$

where $\mathcal{L}(q_v)$ is the left hand side of (3.4). We note that the equation (3.4) is agreeable and consistent with (4.27) if $q_v = q_{vs}$ and $\omega < 0$.

Take now a test function $q_v^b \leqslant q_{vs}$. We see that pointwise

$$(\mathcal{L}(q_v) - f_{q_v}(U) - Fh_{q_v})(q_v^b - q_{vs}) \geqslant 0,$$

in all cases, that is if $q_v^b = q_{vs}$ or $q_v^b < q_{vs}$.

This leads us to the formulation of (3.4)-(3.6) as a quasi-variational inequality : $q_v \in L^\infty(0, t_1; L^2(\mathcal{M})) \cap L^2(0, t_1; H^1(\mathcal{M})), q_v \leqslant q_{vs} = Q_{vs}(p, T)$ and

$$\int_0^{t_1} \left[\langle \partial_t q_v, q_v^b - q_v \rangle + a_{q_v}(q_v, q_v^b - q_v) + b(\mathbf{u}, q_v, q_v^b - q_v) - l_{q_v}(q_v^b - q_v) \right] dt \\ \geqslant \int_0^{t_1} (f_{q_v}(U) + Fh_{q_v}, q_v^b - q_v) dt, \tag{4.28}$$

for all $q_v^b \in L^\infty(0, t_1; H^1)$ with $q_v^b \leqslant q_{vs} = Q_{vs}(p, T)$.

In addition,

$$q_v(t = 0) = q_{v0}. \tag{4.29}$$

At this point, let us introduce what we will call here a solution of (4.21) in the weak sense. Let $U_0 \in \mathbb{V}$ be such that $0 \leqslant q_{v0} \leqslant q_{s0}$ and let $t_1 > 0$ be an arbitrary but fixed constant. A vector $U = U(t) = (q_v, \bar{U}) \in L^2(0, t_1; \mathcal{K}) \cap C([0, t_1]; \mathbb{V})$ with $\partial_t \bar{U} \in L^2(0, t_1; (V^3)^*)$, $\partial_t q_v \in L^{5/3}(0, t_1; V^*)$ is a solution to the initial-boundary value problem (4.21)-(4.23)-(4.24), if, for almost every $t \in [0, t_1]$ and for every $U^b \in \mathcal{K}$, we have (4.25) and (4.28) satisfied.

We recall here that q_{vs} is given by (2.8)-(2.10).

4.3. **The penalized and regularized problem.** To deal with the inequality constraint $q_v \leqslant q_{vs}$ and the discontinuity of the Heaviside function $\mathcal{H}$, we introduce a penalized and regularized version of the problem associated with the parameters $\varepsilon_1, \varepsilon_2 > 0$. The penalization is introduced below by introduction of the term ${\varepsilon_1}^{-1}((q_v - q_{vs})^+)^{3/2}$. We address the discontinuity of the Heaviside function as in [CFTT13] and [CT12]. Recall the multi-valued Heaviside function

$$\mathcal{H}(r) = \begin{cases} 0 & \mathit{for}\ r < 0, \\ [0, 1] & \mathit{for}\ r = 0, \\ 1 & \mathit{for}\ r > 0, \end{cases} \tag{4.30}$$

and the single-valued function h_{q_v} where $h_{q_v} \in \mathcal{H}(q_v - q_{vs})$. Following [TWu15], we can characterize $h_{q_v} \in \mathcal{H}(q_v - q_{vs})$ by

$$([q_v^b - q_{vs}]^+, 1) - ([q_v - q_{vs}]^+, 1) \geqslant \langle h_{q_v}, q_v^b - q_v \rangle \ \text{ for a.e. } t \in [0, t_1],\ \forall q_v^b \in V. \tag{4.31}$$

Now we approximate h_{q_v} by $\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$ for $\varepsilon_2 > 0$, where $\mathcal{H}_{\varepsilon_2}(r)$ is defined as

$$\mathcal{H}_{\varepsilon_2}(r) = \begin{cases} 0 & for\ r \leqslant 0, \\ r/\varepsilon_2 & for\ r \in (0, \varepsilon_2], \\ 1 & for\ r > \varepsilon_2. \end{cases} \tag{4.32}$$

In this setting, Fh_{q_v} ($\sim F\mathcal{H}$) in the right hand side of (3.4) and (4.28) are replaced by $F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$. Similarly, the h_{q_v} ($\sim \mathcal{H}$) in the equations for θ', q_c and q_r are replaced by $\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$ as well. Here the regularized $f(U) + \mathcal{F}\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$ has the same boundedness as the original one. Now the related penalized and regularized system of equations reads

$$\begin{cases} \partial_t q_v + \mathcal{A}_v q_v + \mathbf{v} \cdot \nabla q_v + \omega \frac{\partial q_v}{\partial p} + \frac{1}{\varepsilon_1}((q_v - q_{vs})^+)^{3/2} = f_{q_v}(U) + F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), \\ \partial_t q_c + \mathcal{A}_c q_c + \mathbf{v} \cdot \nabla q_c + \omega \frac{\partial q_c}{\partial p} = f_{q_c}(U) - F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), \\ \partial_t q_r + \mathcal{A}_r q_r + \mathbf{v} \cdot \nabla q_r + \omega \frac{\partial q_r}{\partial p} = f_{q_r}(U), \\ \partial_t \theta' + \mathcal{A}_\theta \theta' + \mathbf{v} \cdot \nabla \theta' + \omega \frac{\partial \theta'}{\partial p} = f_{\theta'}(U) - \frac{L}{c_p \Pi} F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}). \end{cases} \tag{4.33}$$

It is worth noticing that the unknown functions above depend on the small parameter $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Therefore, we will use the notation $U^\varepsilon = (q_v^\varepsilon, q_c^\varepsilon, q_r^\varepsilon, \theta'^\varepsilon)$, etc. The initial and boundary conditions associated with (4.33) are the same for U:

$$U^\varepsilon(x, y, p, 0) = U_0(x, y, p), \tag{4.34}$$

$$\partial_p U^\varepsilon = \mathcal{C}(U_* - U) \text{ on } \Gamma_i, \quad \partial_{n_\mathcal{A}} U^\varepsilon = 0 \text{ on } \Gamma_u \cup \Gamma_l. \tag{4.35}$$

To reveal the structural properties of the systems (4.21) and (4.33), we first consider the products related to the nonlinearity $f(U) + \mathcal{F}\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$, i.e., the following quantities

$$(f_{q_c}(U) - F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), q_c),\ (f_{q_v}(U) + F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), q_v),$$

$$(f_{q_r}(U), q_r),\ (f_{\theta'}(U) - \frac{L}{c_p \Pi} F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), \theta'),$$

where $U \in \mathbb{V}$. By analogy with (4.25) the weak formulation of this problem is to find a function $U^\varepsilon = U^\varepsilon(t) = (q_v^\varepsilon, q_c^\varepsilon, q_r^\varepsilon, \theta'^\varepsilon) \in L^2(0, t_1; \mathbb{V})$ with $\partial_t \bar{U}^\varepsilon \in L^2(0, t_1; (V^3)^*)$ and $\partial_t q_v^\varepsilon \in L^{5/3}(0, t_1; V^*)$, such that

$$\int_0^{t_1} [\langle \partial_t U^\varepsilon, U^b \rangle + a(U^\varepsilon, U^b) + b(\mathbf{u}, U^\varepsilon, U^b) - l(U^b) + \frac{1}{\varepsilon_1}\langle ((q_v^\varepsilon - q_s^\varepsilon)^+)^{3/2}, q_v^b \rangle] dt$$
$$= \int_0^{t_1} (f(U^\varepsilon) + \mathcal{F}\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), U^b) dt, \tag{4.36}$$

for all $\bar{U}^b \in L^2(0, t_1; V^3)$ and $q_v^b \in L^\infty(0, t_1; V)$.

$$U^\varepsilon(0) = U_0. \tag{4.37}$$

4.4. **The formal a priori estimates and existence of solution for** (4.36)**.** We set $U^b = U^\varepsilon$ in (4.36) and we deduce a new energy equality which is in fact obtained by adding the corresponding energy equalities for each component of U^ε, namely $q_v^\varepsilon, q_c^\varepsilon, q_r^\varepsilon$ and θ'^ε. For example, the energy equality for q_c is obtained by multiplying $(4.33)_2$ by q_c and integrating over $\mathcal{M}$, etc.

From now, aiming to simplify the presentation, we will omit the dependence on ε of U^ε that we will denote instead by U; the superscript ε will be reintroduced when it is necessary. Hence, for q_c, using (4.16)–(4.18), we obtain

$$\frac{1}{2}\frac{d}{dt}|q_c|^2_{L^2} + (\mathcal{A}_{q_c}q_c, q_c) + (\mathbf{v}\cdot\nabla q_c, q_c) + (\omega\frac{\partial q_c}{\partial p}, q_c) = (f_{q_c}(U) - F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), q_c). \quad (4.38)$$

The other energy equations can be treated similarly except for the RHS of the analogous equations similar to (4.38).

Now, we start by computing the terms in the LHS of (4.38). Hence, using the definition of $\mathcal{A}_{q_c}$ as in (3.2) and integrating by parts, we deduce that

$$\begin{aligned}(\mathcal{A}_{q_c}q_c, q_c) &= \Big(-\mu_{q_c}\Delta q_c - \nu_{q_c}\partial_p\big((\frac{gp}{R\bar{\theta}})^2\partial_p\big)q_c, q_c\Big)\\ &= \mu_{q_c}|\nabla q_c|^2_{L^2} + \mu_{q_c}\int_{\Gamma_l}\frac{\partial q_c}{\partial n}q_c d\Gamma_l + \nu_{q_c}\Big|\frac{gp}{R\bar{\theta}}\frac{\partial q_c}{\partial p}\Big|^2\\ &\quad + \nu_{q_c}\int_{\Gamma_u}(\frac{gp}{R\bar{\theta}})^2\partial_p q_c\, q_c d\Gamma_u - \nu_{q_c}\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2\partial_p q_c\, q_c d\Gamma_i.\end{aligned} \quad (4.39)$$

Then, we simply observe that, thanks to (3.12), the second and fourth terms in the RHS of (4.39) vanish. We use again the boundary conditions (3.12) to replace the value of the last term in the RHS of (4.39), and we infer that

$$\begin{aligned}(\mathcal{A}_{q_c}q_c, q_c) &= \mu_{q_c}|\nabla q_c|^2_{L^2} + \nu_{q_c}\Big|\frac{gp}{R\bar{\theta}}\frac{\partial q_c}{\partial p}\Big|^2_{L^2}\\ &\quad + \nu_{q_c}\beta_c\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_c^2 d\Gamma_i - \nu_{q_c}\beta_c\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_c\, q_{c*} d\Gamma_i.\end{aligned} \quad (4.40)$$

The second term in the LHS of (4.38) can be computed using again the integration by parts formula. Hence we have

$$\begin{aligned}(\mathbf{v}\cdot\nabla q_c, q_c) &= -\frac{1}{2}\int_{\mathcal{M}}\operatorname{div}\mathbf{v}\; q_c^2 d\mathcal{M} - \frac{1}{2}\int_{\Gamma_l}(q_c)^2\mathbf{v}\cdot\mathbf{n}\, d\Gamma_l\\ &= (\text{since } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\mathcal{M} \text{ by definition of } \mathbf{H})\\ &= -\frac{1}{2}\int_{\mathcal{M}}\operatorname{div}\mathbf{v}\; q_c^2 d\mathcal{M}.\end{aligned} \quad (4.41)$$

We now calculate the last term in the LHS of (4.38) which reads as follows:

$$(\omega\frac{\partial q_c}{\partial p}, q_c) = -\int_{\mathcal{M}}\frac{\partial\omega}{\partial p}q_c^2 d\mathcal{M} - \int_{\mathcal{M}}\omega\frac{\partial q_c}{\partial p}q_c d\mathcal{M},$$

and this yields

$$(\omega\frac{\partial q_c}{\partial p}, q_c) = -\frac{1}{2}\int_{\mathcal{M}} \frac{\partial \omega}{\partial p} q_c^2 d\mathcal{M}. \tag{4.42}$$

We recall here that the velocity $\mathbf{u} = (\mathbf{v}, \omega)$ satisfies $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\mathcal{M}$ and $\nabla\cdot\mathbf{u} = \operatorname{div}\mathbf{v} + \partial\omega/\partial p = 0$ in $\mathcal{M}$. This implies in particular that

$$(\mathbf{v}\cdot\nabla q_c, q_c) + (\omega\frac{\partial q_c}{\partial p}, q_c) = 0. \tag{4.43}$$

Now, combining (4.40) and (4.43) in (4.38), we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|q_c|_{L^2}^2 + \mu_{q_c}|\nabla q_c|_{L^2}^2 + \nu_{q_c}\left|\frac{gp}{R\bar{\theta}}\frac{\partial q_c}{\partial p}\right|_{L^2}^2 + \nu_{q_c}\beta_c\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_c^2 d\Gamma_i \\ = \nu_{q_c}\beta_c\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_c\, q_{c*} d\Gamma_i + (f_{q_c}(U) - F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), q_c).\end{aligned} \tag{4.44}$$

At this level we are able to estimate the RHS of (4.44) starting by its first term. For that purpose, we use the Cauchy-Schwarz inequality and the identity $2ab \leqslant a^2 + b^2$. We then infer that

$$\nu_{q_c}\beta_c\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2|q_c\, q_{c*}|d\Gamma_i \leqslant \frac{\nu_{q_c}\beta_c}{2}\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_c^2 d\Gamma_i + \frac{\nu_{q_c}\beta_c}{2}\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_{c*}^2 d\Gamma_i. \tag{4.45}$$

Finally, we observe that, on the one hand, a part of the last term in the RHS of (4.44) is negative, and, on the other hand, the remaining part can be handled using some estimates for F and the Cauchy-Schwarz inequality. Indeed, using (3.11), we have

$$(f_{q_c}(U), q_c) = -k_1((q_c - q_{crit})^+, q_c) - k_2(q_c\tau(q_r)^{0.875}, q_c), \tag{4.46}$$

where the RHS of (4.46) are negative since $k_1, k_2 \geqslant 0$, and using the definition of $(q_c - q_{crit})^+$ and of $\tau(q_r)$. Furthermore, we use the fact that $\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$ and F are uniformly bounded (see remark 2.3) to estimate the quantity $(-F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), q_c)$. We deduce that

$$\begin{aligned}|(-F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), q_c)| &\leqslant \mathcal{H}_{\varepsilon_2}(q_v - q_{vs})|F|_\infty|q_c| \\ &\leqslant \kappa_1|q_c|^2 + \kappa_2,\end{aligned} \tag{4.47}$$

where κ and the κ_i are generic constants independent of ε and taking different values at different places.

Therefore, combining (4.45), (4.46) and (4.47) in (4.44) we conclude that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|q_c|_{L^2}^2 + \mu_{q_c}|\nabla q_c|_{L^2}^2 + \nu_{q_c}\left|\frac{gp}{R\bar{\theta}}\frac{\partial q_c}{\partial p}\right|_{L^2}^2 + \frac{\nu_{q_c}\beta_c}{2}\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_c^2 d\Gamma_i \\ +(k_1(q_c - q_{crit})^+ + k_2 q_c|q_r|^{0.875}, q_c) \leqslant \frac{\nu_{q_c}\beta_c}{2}\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_{c*}^2 d\Gamma_i + \kappa_1|q_c|^2 + \kappa_2 \\ \leqslant \kappa_1|q_c|^2 + \kappa_2.\end{aligned} \tag{4.48}$$

Using the Gronwall inequality, we conclude, for $q_c = q_c^\varepsilon$, that

$$|q_c^\varepsilon|_{L^\infty(0,t_1;L^2(\mathcal{M}))} \leqslant \kappa, \tag{4.49}$$

$$|q_c^\varepsilon|_{L^2(0,t_1;H^1(\mathcal{M}))} \leqslant \kappa, \tag{4.50}$$

where κ and the κ_i are constants independent of ε as mentioned above.

The same conclusions as in (4.49)-(4.50) hold for q_v^ε, q_r^ε and θ'^ε. Although the energy identity for q_v^ε contains a penalization term as stated in (4.33), this term does not affect the analysis above because of its positivity, namely we have $(\frac{1}{\varepsilon_1}((q_v - q_{vs})^+)^{3/2}, q_v) \geqslant 0$ since $q_{vs} \geqslant 0$. Nevertheless, the estimates for the terms q_v^ε, $q_r^\varepsilon, \theta'^\varepsilon$ need a slightly different treatment for the corresponding right hand sides as they depend on the function $f(U) + \mathcal{F}\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$ and the corresponding value is different for each component of U.

In the following we will explain how to treat these terms and the emphasis will be on the differences regarding the analysis done above for q_c. First, let us start by the q_v-equation $(4.33)_1$. After taking the inner product of $(4.33)_1$ with q_v, we end with the same equation as (4.44) which reads, using (3.6), as follows

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|q_v|_{L^2}^2 + \mu_{q_v}|\nabla q_v|_{L^2}^2 + \nu_{q_v}\left|\frac{gp}{R\bar{\theta}}\frac{\partial q_v}{\partial p}\right|_{L^2}^2 + \nu_{q_v}\beta_v\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_v^2 d\Gamma_i \\ &+ \frac{1}{\varepsilon_1}\int_{\mathcal{M}}((q_v - q_{vs})^+)^{3/2} q_v d\mathcal{M} = \nu_{q_v}\beta_v\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_v\, q_{v*} d\Gamma_i \\ &+ (F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}) + k_3\tau(q_r)^{0.5}(q_{vs} - q_v)^+, q_v).\end{aligned} \tag{4.51}$$

Using the facts that q_{vs} and $\tau(q_r)$ are bounded as stated in Remarks 2.1 and 2.2, we have

$$|(k_3\tau(q_r)^{0.5}(q_{vs} - q_v)^+, q_v)| \leqslant \kappa_1|q_v|^2 + \kappa_2. \tag{4.52}$$

The other terms in the RHS of (4.51) can be estimated as we did in (4.45) and (4.47), and we conclude that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|q_v|_{L^2}^2 + \mu_{q_v}|\nabla q_v|_{L^2}^2 + \nu_{q_v}\left|\frac{gp}{R\bar{\theta}}\frac{\partial q_v}{\partial p}\right|_{L^2}^2 + \nu_{q_v}\beta_v\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2 q_v^2 d\Gamma_i \\ &\quad + \frac{1}{\varepsilon_1}\int_{\mathcal{M}}((q_v - q_{vs})^+)^{3/2} q_v d\mathcal{M} \leqslant \kappa_1|q_v|^2 + \kappa_2,\end{aligned} \tag{4.53}$$

and the Gronwall Lemma implies the desired estimates, namely

$$|q_v^\varepsilon|_{L^\infty(0,t_1;L^2(\mathcal{M}))} \leqslant \kappa, \tag{4.54}$$

$$|q_v^\varepsilon|_{L^2(0,t_1;H^1(\mathcal{M}))} \leqslant \kappa. \tag{4.55}$$

We also have

$$\begin{aligned}&\frac{1}{\varepsilon_1}\int_0^{t_1}\int_{\mathcal{M}}((q_v - q_{vs})^+)^{3/2} q_v d\mathcal{M}ds = \frac{1}{\varepsilon_1}\int_0^{t_1}\int_{\mathcal{M}}[(q_v - q_{vs})^+]^{5/2} d\mathcal{M}ds + \\ &\frac{1}{\varepsilon_1}\int_0^{t_1}\int_{\mathcal{M}}((q_v - q_{vs})^+)^{3/2} q_{vs} d\mathcal{M}ds \leqslant \kappa,\end{aligned}$$

and since $q_{vs} \geqslant 0$,

$$\frac{1}{\varepsilon_1}\int_0^{t_1}\int_{\mathcal{M}}[(q_v - q_{vs})^+]^{5/2} d\mathcal{M} ds \leqslant \kappa. \tag{4.56}$$

More estimates about the penalization term resulting from (4.53),(4.54) and (4.55) will be deduced below (see Lemma 4.2).

Then, we consider first the equation of θ' and let the equation of q_r to the end since the RHS of its equation depends on $\partial\theta'/\partial p$ as stated in (3.19). Thus we multiply the equation of θ', given by $(4.33)_4$, by θ' and integrate over $\mathcal{M}$. Since the boundary conditions for θ' are the same as those of q_c, we obtain an equation similar to (4.44). Remembering also (3.23), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\theta'|^2_{L^2} + \mu_{\theta'}|\nabla\theta'|^2_{L^2} + \nu_{\theta'}\left|\frac{gp}{R\bar{\theta}}\frac{\partial\theta'}{\partial p}\right|^2_{L^2} + \nu_{\theta'}\alpha\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2\theta'^2 d\Gamma_i = \nu_{\theta'}\alpha\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2\theta'\,\theta'_* d\Gamma_i \\
&\quad + \big(-\frac{\theta_h N_h^2}{g}\omega - \frac{L}{c_p\Pi}\big(F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}) + k_3\tau(q_r)^{0.5}(q_{vs} - q_v)^+\big) + f_\theta^1 + \omega\frac{\partial\theta_h(p)}{\partial p}, \theta'\big) \\
&\quad \leqslant \frac{\nu_{\theta'}\alpha}{2}\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2\theta'^2 d\Gamma_i + \frac{\nu_{\theta'}\alpha}{2}\int_{\Gamma_i}(\frac{gp}{R\bar{\theta}})^2\theta'^2_* d\Gamma_i + \kappa_1|\theta'|^2_{L^2} + \kappa_2.
\end{aligned} \tag{4.57}$$

Indeed, it is easy to see that the second term in the middle equation (4.57) is bounded by $\kappa_1|\theta'|^2_{L^2} + \kappa_2$ since the terms θ_h, ω and $\partial\theta_h(p)/\partial p$ are bounded in $L^\infty(\mathcal{M})$. In particular, (4.57) yields

$$\frac{1}{2}\frac{d}{dt}|\theta'|^2_{L^2} + \mu_{\theta'}|\nabla\theta'|^2_{L^2} + \nu_{\theta'}\left|\frac{gp}{R\bar{\theta}}\frac{\partial\theta'}{\partial p}\right|^2_{L^2} \leqslant \kappa_1|\theta'|^2_{L^2} + \kappa_2. \tag{4.58}$$

As before, the application of the Gronwall Lemma to (4.58) gives the following estimates

$$|\theta'^{\varepsilon}|_{L^\infty(0,t_1;L^2(\mathcal{M}))} \leqslant \kappa, \tag{4.59}$$

$$|\theta'^{\varepsilon}|_{L^2(0,t_1;H^1(\mathcal{M}))} \leqslant \kappa. \tag{4.60}$$

Finally, for the q_r-equation given by $(4.33)_3$, we write the equivalent of (4.44), which is simply obtained by multiplying $(4.33)_3$ by q_r and integrating over $\mathcal{M}$, and we use (3.19). Therefore, using the fact that $\tau(q_r)$ and $\partial\theta'/\partial p$ are bounded independently of ε in $L^2(\mathcal{M})$ for a.e. $t \geqslant 0$, see the definition of $\tau(q_r)$ in the end of Remark **??**, we obtain, with $\theta_{\wedge\alpha} = min(\theta,\alpha)$, see after (3.19):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|q_r|^2_{L^2} + \mu_{q_r}|\nabla q_r|^2_{L^2} + \nu_{q_r}\left|\frac{gp}{R\bar\theta}\frac{\partial q_r}{\partial p}\right|^2_{L^2} + \nu_{q_r}\beta_r\int_{\Gamma_i}(\frac{gp}{R\bar\theta})^2 q_r^2 d\Gamma_i \\
&= \nu_{q_r}\beta_r\int_{\Gamma_i}(\frac{gp}{R\bar\theta})^2 q_r\, q_{r*} d\Gamma_i + \Big(-5.32\, g(\frac{\tau(q_r)^{1.2}}{R\Pi\theta_\alpha} + \frac{1.2p\,\tau(q_r)^{0.2}}{R\Pi\theta_\alpha}\frac{\partial q_r}{\partial p} \\
&- \frac{p\,\tau(q_r)^{1.2}}{R\Pi\theta_\alpha^2}\big(\theta_\alpha\frac{\kappa}{p} + \frac{\partial\theta_h(p)}{\partial p} + \frac{\partial\theta'}{\partial p})\big) \\
&- k_3\tau(q_r)^{0.5}(q_{vs}-q_v)^+ + k_1(q_c-q_{crit})^+ + k_2 q_c\,\tau(q_r)^{0.875}, q_r\Big) \\
&\leqslant \text{(using the fact that } 0\leqslant\tau(q_r)\leqslant 1 \text{ and } \theta_\alpha\leqslant\alpha) \\
&\leqslant \frac{\nu_{q_r}\beta_r}{2}\int_{\Gamma_i}(\frac{gp}{R\bar\theta})^2 q_r^2 d\Gamma_i + \frac{\nu_{q_r}\beta_r}{2}\int_{\Gamma_i}(\frac{gp}{R\bar\theta})^2 q_{r*} d\Gamma_i + \beta|q_v|_{L^2}|q_r|_{L^2} + k_1|q_c|_{L^2}|q_r|_{L^2} \\
&+ \kappa_1|q_r|^2_{L^2} + \kappa_2 + \kappa_1\int_{\mathcal{M}}\{|\frac{\partial q_r}{\partial p}| + |\frac{\partial\theta'}{\partial p}| + |q_c|\}d\mathcal{M}.
\end{aligned} \tag{4.61}$$

As mentioned above, q_c, q_v and $\partial\theta'/\partial p$ are bounded in $L^2(0,t_1;L^2(\mathcal{M}))$ independently of ε. Hence we infer that

$$\frac{1}{2}\frac{d}{dt}|q_r|^2_{L^2} + \mu_{q_r}|\nabla q_r|^2_{L^2} + \nu_{q_r}\left|\frac{gp}{R\bar\theta}\frac{\partial q_r}{\partial p}\right|^2_{L^2} \leqslant \mathcal{G}(t), \tag{4.62}$$

where $\mathcal{G} = \mathcal{G}(t)$ is a generic function of t, bounded in $L^1(0,t_1)$ independently of ε.

Finally it suffices to apply the Gronwall inequality to conclude that,

$$|q_r^\varepsilon|_{L^\infty(0,t_1;L^2(\mathcal{M}))} \leqslant \kappa, \tag{4.63}$$
$$|q_r^\varepsilon|_{L^2(0,t_1;H^1(\mathcal{M}))} \leqslant \kappa. \tag{4.64}$$

4.5. **A priori estimates on the time derivative of U.** We now aim to derive a priori estimates for the time derivatives of U in view of obtaining a strong convergence result for these functions and especially θ' ($\sim T$), by application of a compactness theorem.

More precisely, we prove in this subsection some a priori estimates for the solution U of the system (4.33) associated with the initial and boundary conditions (4.34) and (4.35), respectively. The intent is to show that the time derivative of $\bar U = \bar U^\varepsilon = (q_c^\varepsilon, q_r^\varepsilon, \theta'^\varepsilon)$ and q_v^ε, recalling here the dependence of the solution U on ε, are bounded independently of ε. Therefore, for $\bar U^\varepsilon$, as we did in Subsection 4.4 we will develop here the computations for one component of $\bar U$ and then explain only the differences for the other components. The estimate for the time derivative of q_v^ε is more subtle and will be treated differently later on (see a similar *easier* situation in [TWu15], [TWa15]). More precisely, let us start with the q_c-equation $(4.33)_2$ that we multiply by $\partial q_c/\partial t$ and integrate over $\mathcal{M}$.

Hence, using the symmetry of $\mathcal{A}_c$ and then the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\int_{\mathcal{M}} \left|\frac{\partial q_c}{\partial t}\right|^2 d\mathcal{M} + \frac{1}{2}\frac{d}{dt}(\mathcal{A}_c q_c, q_c) &= \left(f_{q_c}(U) - F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}) - \mathbf{v}\cdot\nabla q_c - \omega\frac{\partial q_c}{\partial p}, \frac{\partial q_c}{\partial t}\right) \\ &\leqslant J_1 + \frac{1}{2}\left|\frac{\partial q_c}{\partial t}\right|^2_{L^2}, \end{aligned} \tag{4.65}$$

where

$$J_1 = c|f_{q_c}(U) - F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}) - \mathbf{v}\cdot\nabla q_c - \omega\frac{\partial q_c}{\partial p}|^2_{L^2} \tag{4.66}$$

for an appropriate constant c.

To bound the term J_1 in the RHS of (4.65), we need first to estimate $|f_{q_c}(U)|_{L^2}$. Let us recall the definition of $f_{q_c}(U)$, which is given by (3.11) and it involves itself the expression of $\tau(q_r)$ stated just after (3.11). Hence, we estimate the terms in the RHS of (3.11), one by one, as follows:

$$|k_1(q_c - q_{crit})^+| \leqslant \kappa(1 + |q_c|_{L^2}), \tag{4.67}$$

$$|k_2 q_c \tau(q_r)| \leqslant \kappa. \tag{4.68}$$

Therefore, using the fact that q_{vs} and $\tau(q_r)$ are bounded in $L^\infty(\mathcal{M}\times(0,t_1))$ and q_c is bounded in $L^\infty(0,t_1;L^2(\mathcal{M}))$, we deduce that

$$|f_{q_c}(U)|_{L^2} \leqslant \kappa(1 + |q_c|_{L^2}) \leqslant \kappa, \quad \forall\, t \in (0,t_1). \tag{4.69}$$

Secondly, as F and $\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$ are bounded in $L^\infty(\mathcal{M}\times(0,t_1))$, we also have

$$|F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})|_{L^2} \leqslant \kappa, \quad \forall\, t \in (0,t_1). \tag{4.70}$$

Then, assuming that $\mathbf{u} \in L^\infty(\mathcal{M}\times(0,t_1))$ and using (4.50), we obtain

$$|\mathbf{v}\cdot\nabla q_c|_{L^2} \leqslant |\mathbf{v}|_{L^\infty(\mathcal{M}\times(0,T))}|\nabla q_c|_{L^2} \leqslant \mathcal{G}(t), \tag{4.71}$$

where we denoted again by $\mathcal{G} = \mathcal{G}(t)$ a generic function of time t, bounded in $L^1(0,t_1)$ independently of ε and we recall that $|\nabla q_c|^2_{L^2}$ has been already bounded in $L^1(0,t_1)$. Using again $\mathbf{u} \in L^\infty(\mathcal{M}\times(0,t_1))$ and (4.50), we infer that

$$|\omega\frac{\partial q_c}{\partial p}|_{L^2} \leqslant \kappa\, |\frac{\partial q_c}{\partial p}|_{L^2} \leqslant \mathcal{G}(t), \tag{4.72}$$

and we recall that $|\partial q_c/\partial p|^2_{L^2}$ has been by now bounded in $L^1(0,t_1)$.

Now we derive similar estimates for the other terms q_r and θ'. For that purpose, we follow the same steps as we did for q_c, and we write the equations analogous to (4.65). Hence, the only difference will be here the estimates of the terms $\|f_{q_r}(U)\|_{L^2}$ and $\|f_{\theta'}(U) - \frac{L}{c_p\Pi}F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})\|_{L^2}$. To do that we make use of the expressions of $f_{q_r}(U)$ and $f_{\theta'}(U) - \frac{L}{c_p\Pi}F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$, given respectively by (3.19) and (3.23). For $f_{q_r}(U)$, we have

$$
\begin{aligned}
&|f_{q_r}(U)|_{L^2} = \\
&\Big| - 5.32\, g\big(\frac{\tau(q_r)^{1.2}}{R\Pi\theta_\alpha} + \frac{1.2p\,\tau(q_r)^{0.2}}{R\Pi\theta_\alpha}\frac{\partial q_r}{\partial p} - \frac{p\,\tau(q_r)^{1.2}}{R\Pi\theta_\alpha^2}\big(\theta_\alpha\frac{\kappa}{p} + \frac{\partial\theta_h(p)}{\partial p} + \frac{\partial\theta'}{\partial p}\big)\big) \\
&- k_3\tau(q_r)^{0.5}(q_{vs}-q_v)^+ + k_1(q_c - q_{crit})^+ + k_2 q_c\tau(q_r)^{0.875}\Big|_{L^2} \\
&\leqslant \kappa_1\left\{\left|\frac{\partial q_r}{\partial p}\right|_{L^2} + \left|\frac{\partial\theta'}{\partial p}\right|_{L^2} + |q_c|_{L^2} + |q_v|_{L^2}\right\} + \kappa_2 \\
&\leqslant \mathcal{G}(t) + \left|\frac{\partial q_r}{\partial p}\right|_{L^2}^2 + \left|\frac{\partial\theta'}{\partial p}\right|_{L^2}^2 + |q_c|_{L^2}^2 + |q_v|_{L^2}^2,
\end{aligned}
\tag{4.73}
$$

and we already showed that $|\partial q_r/\partial p|_{L^2}^2$, $|\partial\theta'/\partial p|_{L^2}^2$, $|q_c|_{L^2}^2$, and $|q_v|_{L^2}^2$ are bounded in $L^1(0,t_1)$ thanks to (4.64), (4.60), (4.49) and (4.54).

Similarly, for $f_{\theta'}(U) - \frac{L}{c_p\Pi}F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$, we assume that $\mathbf{u} \in L^\infty(\mathcal{M}\times(0,t_1))$ and we obtain

$$
|f_{\theta'}(U) - \frac{L}{c_p\Pi}F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})|_{L^2} \leqslant \mathcal{G}(t) + |q_v|_{L^2}^2. \tag{4.74}
$$

Therefore we conclude that $J_1 \leqslant \mathcal{G}(t)$ for all $t \in (0,t_1)$, and we infer that

$$
|\partial_t q_c^\varepsilon|_{L^2(0,t_1;L^2(\mathcal{M}))} \leqslant \kappa; \tag{4.75}
$$

here κ is a constant independent of ε and t.

Now we take the $L^2(\mathcal{M})$ inner product of $\mathcal{A}_c q_c$ with $(4.33)_2$, and we can apply a similar argument as what we did for $\partial_t q_c$ and obtain

$$
|\mathcal{A}_c q_c|_{L^2(0,t_1;L^2)} \leqslant \kappa. \tag{4.76}
$$

Consequently, using (4.50), (4.75), (4.76) and $(4.33)_2$, we deduce that

$$
\frac{\partial q_c^\varepsilon}{\partial t} \text{ and } \mathcal{A}_c q_c^\varepsilon \text{ are bounded in } L^2(0,t_1;L^2), \text{ independently of } \varepsilon, \tag{4.77}
$$

that is q_c is bounded in $L^2(0,t_1;H^2)$ independently of ε.

Similar estimates to (4.75) follow in a straightforward manner for q_r and θ'.

Before we move on to bound the time derivative of q_v, we add one more estimate on $\bar{U}$. Integrating (4.65) on $(0,t)$ for any $t \in [0,t_1]$, we have

$$
\frac{1}{2}\int_0^t \left|\frac{\partial q_c}{\partial t}\right|_{L^2}^2 dt + (\mathcal{A}_c q_c(t), q_c(t)) \leqslant \int_0^{t_1} J_1 dt + (\mathcal{A}_c q_{c0}, q_{c0}) \leqslant \kappa, \tag{4.78}
$$

where J_1 was defined in (4.66). Again, the constant κ is independent of ε and t. So $(\mathcal{A}_c q_c(t), q_c(t))$ is bounded uniformly in time for any $t \in [0,t_1]$. This implies

$$
|q_c^\varepsilon|_{L^\infty(0,t_1;H^1(\mathcal{M}))} \leqslant \kappa. \tag{4.79}
$$

Estimates similar to (4.79) hold for q_r and θ'. In particular, we will use the bound

$$|\theta'^{\varepsilon}|_{L^\infty(0,t_1;H^1(\mathcal{M}))} \leqslant \kappa \tag{4.80}$$

in the estimate of the time derivative of q_v^ε.

For q_v^ε, we will show its time derivative is bounded independently of ε in $L^{5/3}(0,t_1;V^*)$. The main issue here is to control the penalization term which contains the "large" factor $\frac{1}{\varepsilon_1}$. We begin with Lemma 4.2 .

Lemma 4.2. *The following bound holds:*

$$\frac{1}{\varepsilon_1^{5/3}} \int_0^{t_1} |(q_v^\varepsilon - q_{vs}^\varepsilon)^+|^{5/2}_{L^{5/2}(\mathcal{M})} dt \leqslant \kappa, \tag{4.81}$$

where κ is a constant independent of ε.

Proof. We multiply $(4.33)_1$ by $(q_v - q_{vs})^+$ and integrate on $\mathcal{M}$, we find

$$\begin{aligned}&\left(\frac{\partial q_v}{\partial t}, (q_v - q_{vs})^+\right) + (\mathcal{A}_v q_v, (q_v - q_{vs})^+) + \frac{1}{\varepsilon_1}\int_{\mathcal{M}} ((q_v - q_{vs})+)^{5/2} d\mathcal{M} \\ &= \left(f_{q_v}(U) + F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}) - \mathbf{v}\cdot\nabla q_v - \omega\frac{\partial q_v}{\partial p}, (q_v - q_{vs})^+\right).\end{aligned} \tag{4.82}$$

The first two terms in the LHS can be rewritten as

$$\begin{aligned}(\partial_t q_v, (q_v - q_{vs})^+) &= (\partial_t(q_v - q_{vs}), (q_v - q_{vs})^+) + (\partial_t q_{vs}, (q_v - q_{vs})^+) \\ &= \frac{1}{2}\frac{d}{dt}|(q_v - q_{vs})^+|^2_{L^2(\mathcal{M})} + (\partial_t q_{vs}, (q_v - q_{vs})^+), \\ (\mathcal{A}_v q_v, (q_v - q_{vs})^+) &= (\mathcal{A}_v(q_v - q_{vs}), (q_v - q_{vs})^+) + (\mathcal{A}_v q_{vs}, (q_v - q_{vs})^+) \\ &= (\mathcal{A}_v(q_v - q_{vs})^+, (q_v - q_{vs})^+) + (\mathcal{A}_v q_{vs}, (q_v - q_{vs})^+).\end{aligned}$$

Dropping the positive term: $(\mathcal{A}_v(q_v - q_{vs})^+, (q_v - q_{vs})^+)$ in the LHS, we can deduce from (4.82) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|(q_v - q_{vs})^+|^2_{L^2} + \frac{1}{\varepsilon_1}|((q_v - q_{vs})+)|^{5/2}_{L^{5/2}} \\ &\leqslant |(\mathcal{A}_v q_{vs}, (q_v - q_{vs})^+) + (\partial_t q_{vs}, (q_v - q_{vs})^+) + (\mathbf{u}\cdot\nabla_3 q_v, (q_v - q_{vs})^+) \\ &- (f_{q_v}(U) + F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})), (q_v - q_{vs})^+)|.\end{aligned} \tag{4.83}$$

Using Hölder and Young inequalities, the RHS of (4.83) can be estimated in the following way:

$$
\begin{aligned}
|(\mathcal{A}_v q_{vs}, (q_v - q_{vs})^+)| &= \left| \int_{\mathcal{M}} \varepsilon_1^{2/5} \mathcal{A}_v q_{vs} \cdot \frac{(q_v - q_{vs})^+}{\varepsilon_1^{2/5}} d\mathcal{M} \right| \\
&\leqslant |\varepsilon_1^{2/5} \mathcal{A}_v q_{vs}|_{L^{5/3}} \left| \frac{(q_v - q_{vs})^+}{\varepsilon_1^{2/5}} \right|_{L^{5/2}} \\
&\leqslant C \varepsilon_1^{2/3} |\mathcal{A}_v q_{vs}|_{L^{5/3}}^{5/3} + \frac{1}{8\varepsilon_1} |(q_v - q_{vs})^+|_{L^{5/2}}^{5/2}.
\end{aligned} \tag{4.84}
$$

The other terms can be addressed similarly. Then (4.83) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|(q_v - q_{vs})^+|_{L^2}^2 + \frac{1}{\varepsilon_1}|(q_v - q_{vs})^+|_{L^{5/2}}^{5/2} \\
&\leqslant C\varepsilon_1^{2/3}(|\mathcal{A}_v q_{vs}|_{L^{5/3}}^{5/3} + |\partial_t q_{vs}|_{L^{5/3}}^{5/3} + |\nabla_3 q_v|_{L^{5/3}}^{5/3} + |q_v|_{L^{5/3}}^{5/3} \\
&+ C_1) + \frac{1}{2\varepsilon_1}|(q_v - q_{vs})^+|_{L^{5/2}}^{5/2}.
\end{aligned} \tag{4.85}
$$

Here $|q_v|_{L^{5/3}}^{5/3} + C_1$ in the RHS of (4.85) is the bound for $|f_{q_v}(U) + F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})|_{L^{5/3}}^{5/3}$.

Integrating now (4.85) in time on $(0, t_1)$, we have

$$
\begin{aligned}
&\frac{1}{2}|(q_v(t_1) - q_{vs}(t_1))^+|_{L^2}^2 - \frac{1}{2}|(q_{v0} - q_{s0})^+|_{L^2}^2 + \frac{1}{2\varepsilon_1}\int_0^{t_1} |(q_v - q_{vs})^+|_{L^{5/2}}^{5/2} dt \\
&\leqslant C\varepsilon_1^{2/3} \int_0^{t_1} (|\mathcal{A}_v q_{vs}|_{L^{5/3}}^{5/3} + |\partial_t q_{vs}|_{L^{5/3}}^{5/3} + |\nabla_3 q_v|_{L^{5/3}(\mathcal{M})}^{5/3} + |q_v|_{L^{5/3}}^{5/3} + C_1) dt.
\end{aligned} \tag{4.86}
$$

The first term in the LHS of (4.86) is positive and the second term is 0 because of the constraint on the initial value $q_{v0} \leqslant q_{s0}$.

To reach the desired bound (4.81) on the penalization term, we will bound the integral in the RHS of (4.86) independently of ε, drop the positive term in the LHS and divide both sides of (4.86) by $\varepsilon_1^{2/3}$. We now estimate each term in the RHS of (4.86).

Both $|q_v|$ and $|\nabla_3 q_v|$ are bounded in $L^{5/3}((0, t_1) \times \mathcal{M})$, thanks to (4.54),(4.55) and the fact that $L^2((0, t_1) \times \mathcal{M}) \subset L^{5/3}((0, t_1) \times \mathcal{M})$.

Then for $\partial_t q_{vs}$, we see that $\partial_t q_{vs} = \frac{\partial Q_{vs}}{\partial T}(p, T) \cdot \partial_t T$. Because of the relationship between T (resp. T') and θ (resp. θ') and recalling that $\partial_t \theta'$ has already been bounded in $L^2((0, t_1) \times \mathcal{M})$, $\partial_t T$ is bounded in $L^2((0, t_1) \times \mathcal{M})$. Also, $\frac{\partial Q_{vs}}{\partial T}(p, T)$ is uniformly bounded by Remark 2.1. Thus we have $|\partial_t q_{vs}|$ bounded in $L^{5/3}((0, t_1) \times \mathcal{M})$.

The most problematic term is $|\mathcal{A}_v q_{vs}|_{L^{5/3}}^{5/3}$. We begin by exploring the relationship between $\Delta_3 q_{vs}$ and T. By the expressions (2.7) and (2.8),

$$\begin{aligned}\frac{\partial^2 q_{vs}}{\partial x^2} &= \frac{\partial}{\partial x}\big(\frac{\partial Q_{vs}(p,T)}{\partial T}\frac{\partial T}{\partial x}\big) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial Q_{vs}(p,T)}{\partial T}\right)\frac{\partial T}{\partial x} + \frac{\partial Q_{vs}(p,T)}{\partial T}\frac{\partial^2 T}{\partial x^2}.\end{aligned} \tag{4.87}$$

Use the fact that $\partial q_{vs}(p,T)/\partial T$ is uniformly bounded as stated in Remark 2.1, we can easily deduce that

$$\Big|\frac{\partial}{\partial x}\big(\frac{\partial Q_{vs}(p,T)}{\partial T}\big)\Big| \leqslant C\Big|\frac{\partial T}{\partial x}\Big|,$$

for some generic constant C that does not depend on ε.

Recalling (4.87), we can further deduce that, pointwisely,

$$\Big|\frac{\partial^2 q_{vs}}{\partial x^2}\Big| \leqslant C\left(\Big|\frac{\partial T}{\partial x}\Big|^2 + \Big|\frac{\partial^2 T}{\partial x^2}\Big|\right).$$

Similarly,

$$\Big|\frac{\partial^2 q_{vs}}{\partial y^2}\Big| \leqslant C\left(\Big|\frac{\partial T}{\partial y}\Big|^2 + \Big|\frac{\partial^2 T}{\partial y^2}\Big|\right).$$

The second derivative of q_{vs} with respect to p is slightly different with $\partial^2 q_{vs}/\partial x^2$ and $\partial^2 q_{vs}/\partial y^2$, as q_{vs} depends on p explicitly.

By (2.8), it can be easily calculated that

$$\frac{\partial q_{vs}}{\partial p} = \frac{1}{(p - 0.378 e_{vs})}\left(\frac{273ap}{T^2}\cdot\frac{\partial T}{\partial p} - 1\right) q_{vs}.$$

Differentiating with respect to p one more time, we can deduce that, after some algebra:

$$\Big|\frac{\partial^2 q_{vs}}{\partial p^2}\Big| \leqslant C\left(\Big|\frac{\partial T}{\partial p}\Big| + \Big|\frac{\partial T}{\partial p}\Big|^2 + \Big|\frac{\partial^2 T}{\partial p^2}\Big| + C_1\right).$$

It follows that

$$|\mathcal{A}_v q_{vs}|^{5/3}_{L^{5/3}(\mathcal{M})} \leqslant C\left(|\Delta_3 T|^{5/3}_{L^{5/3}(\mathcal{M})} + |\nabla_3 T|^{10/3}_{L^{10/3}(\mathcal{M})} + C_1\right). \tag{4.88}$$

By Gagliardo-Nirenberg's interpolation inequality, we have

$$|\nabla_3 T|^{10/3}_{L^{10/3}(\mathcal{M})} \leqslant C\left(|\nabla_3 T|^{10/3}_{L^2(\mathcal{M})} + |\nabla_3 T|^{4/3}_{L^2(\mathcal{M})}|\Delta_3 T|^2_{L^2(\mathcal{M})}\right). \tag{4.89}$$

Note that we also have $\nabla_3 T \in L^\infty(0,t_1;H)$ and $\Delta_3 T \in L^2((0,t_1)\times\mathcal{M})$ by (4.76) and (4.80), and thus

$$\begin{aligned}\int_0^{t_1} |\mathcal{A}_v q_{vs}|^{5/3}_{L^{5/3}(\mathcal{M})} dt &\leqslant C\int_0^{t_1} (|\Delta_3 T|^{5/3}_{L^{5/3}} + |\nabla_3 T|^{10/3}_{L^{10/3}} + C_1) dt \\ &\leqslant C\int_0^{t_1} (|\Delta_3 T|^{5/3}_{L^{5/3}} + |\nabla_3 T|^{10/3}_{L^2} + |\nabla_3 T|^{4/3}_{L^2}|\Delta_3 T|^2_{L^2} + C_1) dt \\ &\leqslant C\Big(|\Delta_3 T|^{5/3}_{L^{5/3}((0,t_1)\times\mathcal{M})} + (\operatorname*{ess\,sup}_{[0,t_1]} |\nabla_3 T|_{L^2})^{10/3}\cdot t_1 \\ &\quad + (\operatorname*{ess\,sup}_{[0,t_1]} |\nabla_3 T|_{L^2})^{4/3} |\Delta_3 T|^2_{L^2((0,t_1)\times\mathcal{M})} + C_1 t_1\Big) \\ &\leqslant \kappa. \end{aligned} \tag{4.90}$$

By now all the terms in the integral in the RHS of (4.86) have been bounded independently of ε, this finishes the proof of Lemma 4.2. □

With the help of Lemma 4.2, we are ready to estimate the time derivative of q_v. We multiply $(4.33)_1$ by $q_v^b \in L^{5/2}(0,t_1;V)$ and integrate on $\mathcal{M}$:

$$\langle \partial_t q_v, q_v^b\rangle + (\mathcal{A}_v q_v, q_v^b) + (\mathbf{u}\cdot\nabla_3 q_v, q_v^b) + (\frac{1}{\varepsilon_1}((q_v - q_{vs})^+)^{3/2}, q_v^b) = (f^{\varepsilon_2}(U), q_v^b). \tag{4.91}$$

Rearranging (4.91), we have

$$\begin{aligned}|\langle \partial_t q_v, q_v^b\rangle| &= |-a_{q_v}(q_v,q_v^b) - b(\mathbf{u},q_v,q_v^b) - (\frac{1}{\varepsilon_1}((q_v-q_{vs})^+)^{3/2}, q_v^b) + l_{q_v}(q_v^b) + (f^{\varepsilon_2}_{q_v}, q_v^b)| \\ &\leqslant C(\|q_v\|_V + \|\mathbf{u}\|_{\mathbf{V}}\|q_v\|_V + \frac{1}{\varepsilon_1}|(q_v-q_{vs})^+|^{3/2}_{L^{5/2}} + |q_v|_{L^2} + C_1)\|q_v^b\|_V. \end{aligned} \tag{4.92}$$

Here we used the Lemma 4.1 and the fact that

$$\begin{aligned}\frac{1}{\varepsilon_1}\int_{\mathcal{M}} ((q_v-q_{vs})^+)^{3/2} q_v^b d\mathcal{M} &\leqslant \frac{1}{\varepsilon_1}|((q_v-q_{vs})^+)^{3/2}|_{L^{5/3}}|q_v^b|_{L^{5/2}} \\ &\leqslant (V\subset L^{5/2}(\mathcal{M}) \text{ in } \mathbb{R}^3) \\ &\leqslant \frac{1}{\varepsilon_1}|(q_v-q_{vs})^+|^{3/2}_{L^{5/2}}\|q_v^b\|_V. \end{aligned} \tag{4.93}$$

Hence,

$$\|\partial_t q_v\|_{V*} \leqslant C(\|q_v\|_V + \|\mathbf{u}\|_{\mathbf{V}}\|q_v\|_V + \frac{1}{\varepsilon_1}|(q_v-q_{vs})^+|^{3/2}_{L^{5/2}} + |q_v|_{L^2} + C_1), \tag{4.94}$$

$$\|\partial_t q_v\|^{5/3}_{V*} \leqslant C(\|q_v\|^{5/3}_V + \|\mathbf{u}\|^{5/3}_{\mathbf{V}}\|q_v\|^{5/3}_V + \frac{1}{\varepsilon_1^{5/3}}|(q_v-q_{vs})^+|^{5/2}_{L^{5/2}} + |q_v|^{5/3}_{L^2} + C_1). \tag{4.95}$$

Then thanks to (4.54),(4.55) and Lemma 4.2,

$$\int_0^{t_1} \|\partial_t q_v\|^{5/3}_{V*} dt \leqslant \kappa, \tag{4.96}$$

where κ, as before, is a constant independent of ε. So we have bound $\partial_t q_v$ in $L^{5/3}(0,t_1;V^*)$ as desired.

Finally, we summarize all the estimates that we obtained componentwise and write this for the solution U^ε. More precisely, we now have

$$|U^\varepsilon|_{L^\infty(0,t_1;\mathbb{H})} \leqslant \kappa,\ \|U^\varepsilon\|_{L^2(0,t_1;\mathbb{V})} \leqslant \kappa,\ \|\bar{U}^\varepsilon\|_{L^2(0,t_1;H^2(\mathcal{M})^3))} \leqslant \kappa,\ \|\partial_t\bar{U}^\varepsilon\|_{L^2(0,t_1;L^2(\mathcal{M})^3)} \leqslant \kappa,$$
$$\|\bar{U}^\varepsilon\|_{L^\infty(0,t_1;H^1(\mathcal{M})^3))} \leqslant \kappa,\ \text{and}\quad |\partial_t q_v^\varepsilon|_{L^{5/3}(0,t_1;V^*)} \leqslant \kappa. \tag{4.97}$$

Remark 4.3. As usual by implementing a Galerkin approximation for the problem (4.33)-(4.37) we can obtain a priori estimates similar to the above estimates for the Galerkin approximation. Then passing to the lower limit we obtain these very estimates (independent of ε) for the actual solution U of (4.33)-(4.37). We state this existence result in the following theorem, but we will skip the proof since it is straightforward after the analysis above on the a priori estimates.

Theorem 4.4. *Let $\varepsilon > 0$ be fixed and assume that $\mathbf{u} \in L^\infty((0,t_1)\times\mathcal{M})$ and $U_0 \in \mathbb{V}$ are given. Then, the system (4.33) associated with the initial and boundary conditions (4.34) and (4.35), respectively, has a solution U^ε such that*

$$U^\varepsilon \in L^\infty(0,t_1;\mathbb{H}) \cap L^2(0,t_1;\mathbb{V}), \tag{4.98}$$

and

$$\bar{U}^\varepsilon \in L^2(0,t_1;H^2),\quad \partial_t\bar{U}^\varepsilon \in L^2(0,t_1;L^2),\quad \partial_t q_v^\varepsilon \in L^{5/3}(0,t_1;V^*). \tag{4.99}$$

Furthermore the norms of $U^\varepsilon, \bar{U}^\varepsilon$ and $\partial_t\bar{U}^\varepsilon$ in the corresponding spaces are bounded independently of ε by quantities which depend on the norm of U_0 in $\mathbb{H}$ and on the other data.

4.6. **Passage to the limit.** In the following we will pass to the limit, as $\varepsilon \to 0$, in the penalized system (4.33), and to avoid a possible confusion we reintroduce here the dependence on ε. First, using (4.97) and Aubin-Lions compactness theorem, we deduce the existence of a subsequence, still denoted $U^\varepsilon = (q_v^\varepsilon, q_c^\varepsilon, q_r^\varepsilon, \theta'^\varepsilon)$, and a function $U = (q_v, q_c, q_r, \theta')$ both verifying (4.98), (4.99), such that, as $\varepsilon \to 0$,

(i) $U^\varepsilon \rightharpoonup U$ weakly in $L^2(0,t_1;\mathbb{V})$ and weak-$*$ in $L^\infty(0,t_1;\mathbb{H})$,
(ii) $\partial_t\bar{U}^\varepsilon \rightharpoonup \partial_t\bar{U}$ weakly in $L^2(0,t_1;L^2(\mathcal{M})^3)$,
(iii) $\partial_t q_v^\varepsilon \rightharpoonup \partial_t q_v$ weakly in $L^{5/3}(0,t_1;V^*)$,
(iv) $\bar{U}^\varepsilon \rightharpoonup \bar{U}$ strongly in $L^2(0,t_1;H^1)$ and weakly in $L^2(0,t_1;H^2)$,
(v) $q_v^\varepsilon \rightharpoonup q_v$ strongly in $L^2(0,t_1;L^2(\mathcal{M}))$ and weakly in $L^2(0,t_1;H^1)$,
(vi) $(q_v^\varepsilon - q_{vs}^\varepsilon)^+ \to 0$ strongly in $L^{5/2}((0,t_1)\times\mathcal{M})$, thanks to Lemma 4.2,
(vii) $\mathcal{H}_{\varepsilon_2}(q_v^\varepsilon - q_{vs}^\varepsilon) \rightharpoonup h_{q_v}$ weak-$*$ in $L^\infty((0,t_1)\times\mathcal{M})$ for $h_{q_v} \in \mathcal{H}(q_v - q_{vs})$,

In view of (i) and (iii), we also have

$$q_v^\varepsilon(t_1) \rightharpoonup q_v(t_1) \text{ weakly in } L^2(\mathcal{M}). \tag{4.100}$$

For the inequality constraint on q_v, after showing that $q_{vs}^\varepsilon \to q_{vs} = Q_{vs}(p,T)$ in $L^2(0,t_1;V)$ (see Lemma 4.5 below), (vi) implies in particular that $q_v \leqslant q_{vs}$.

It is worth noting here that the strong convergence in $L^2(0,t_1;\mathbb{H})$ is in fact available in $L^p(0,t_1;\mathbb{H})$, for all $p \geqslant 1$, thanks to the continuity of $U^\varepsilon \in \mathcal{C}([0,t_1];\mathbb{H})$.

By the continuity and boundedness of $f(U)$, using (iv) and (v), we have

$$f(U^\varepsilon) \rightharpoonup f(U) \text{ strongly in } L^2(0,t_1;L^2(\mathcal{M})). \tag{4.101}$$

where $f(U)$ is seen componentwise as $f(U) = (f_{q_v}(U), f_{q_c}(U), f_{q_r}(U), f_{\theta'}(U))$ and $\mathcal{F}$ represents the vector $(F, -F, 0, -\frac{L}{c_p\Pi}F)$.

Moreover, thanks to the estimates showing the boundedness of $\mathcal{F}\mathcal{H}_{\varepsilon_2}(q_v - q_{vs})$as performed above, using (vii) and [Lio76, Lemma 1.3], we have

$$\mathcal{F}(p, T^\varepsilon)\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}) \rightharpoonup \mathcal{F}(p,T)h_{q_v} \text{ weakly in } L^2(0,t_1;L^2(\mathcal{M})), \tag{4.102}$$

Therefore one can pass to the limit, as $\varepsilon \to 0$, in $(4.33)_{2,3,4}$ (remember here the dependence of the solutions on ε), see [TWa15] and [TWu15].

Moreover, we need the following results which will be used in the proof of the convergence of the penalized term, namely $(4.33)_1$.

Lemma 4.5. *If T^ε converges to T strongly in $L^2(0,t_1;V)$, then $q_{vs}^\varepsilon = Q_{vs}(p,T^\varepsilon)$, as given by (2.7), converges to $q_{vs} = Q_{vs}(p,T)$ strongly in $L^2(0,t_1;V)$.*

Proof. By the expressions (2.7)-(2.9) and Remark 2.1, we see that $q_{vs}^\varepsilon = Q_{vs}(p,T^\varepsilon)$ converges to $q_{vs} = Q_{vs}(p,T)$ in $L^2(0,t_1;L^2)$. Indeed, on the one hand, we recall here the relationship between T (resp. T' and T'^ε) and θ (resp. θ' and θ'^ε) thanks to e.g. (2.12), and on the other hand, we use the fact that $\bar{U}^\varepsilon$ converges to $\bar{U}$ strongly in $(L^2(0,t_1;V))^3$. Similarly since the derivative of Q_{vs} with respect to T is uniformly bounded, thanks to 2.1, we see that $\nabla_{\mathbf{x}} q_{vs}^\varepsilon = \frac{\partial Q_{vs}}{\partial T}(p,T^\varepsilon)\cdot\nabla_{\mathbf{x}}T^\varepsilon$ converges to $\nabla_{\mathbf{x}} q_{vs} = \frac{\partial Q_{vs}}{\partial T}(p,T)\cdot\nabla_{\mathbf{x}}T$ in $L^2((0,t_1)\times\mathcal{M})$.
We thus conclude that q_{vs}^ε converges to q_{vs} strongly in $L^2(0,t_1;V)$. □

Lemma 4.6. *For all $q_v^b \in \mathcal{K} = \mathcal{K}(U)$, we consider $q_v^{b\varepsilon} = q_v^b - (q_v^b - q_{vs}^\varepsilon)^+ = \min(q_v^b, q_{vs}^\varepsilon)$. Then $q_v^{b\varepsilon}$ converges to q_v^b strongly in $L^2(0,t_1;V)$.*

Proof. We first observe, using the definitions of $q_v^{b\varepsilon}$ and of the set $\mathcal{K}$, that $q_v^{b\varepsilon}$ converges to q_v^b in $L^2(0,t_1;L^2)$. Then we see that the derivative of $q_v^{b\varepsilon}$ with respect to the space variable $\mathbf{x}$ can be written as $\nabla_{\mathbf{x}} q_v^{b\varepsilon} = \nabla_{\mathbf{x}} q_v^b - 1_{\{q_v^b > q_{vs}^\varepsilon\}}\nabla_{\mathbf{x}}(q_v^b - q_{vs}^\varepsilon)$. Using Lemma 4.5 we deduce that $q_v^{b\varepsilon}$ converges to q_v^b strongly in $L^2(0,t_1;V)$. □

Remark 4.7. From the proof of Lemma 4.6, we see that $|\nabla_3 q_{vs}^\varepsilon| \leqslant C|\nabla_3 T^\varepsilon| + C_1$. Then noting that $\nabla_3 T^\varepsilon \in L^\infty(0,t_1;L^2(\mathcal{M}))$ by (4.80), here q_{vs}^ε actually lies in a bounded set of $L^\infty(0,t_1;V)$. And by our assumption, $q_v^{b\varepsilon} \in L^\infty(0,t_1;V)$. Hence, $q_v^{b\varepsilon} = \min(q_v^b, q_{vs}^\varepsilon)$ lies in a bounded set in $L^\infty(0,t_1;V)$ as well. Also $q_v^{b\varepsilon}$ converges to q_v^b almost everywhere in V for $t \in [0,t_1]$. Lemma 4.6 together with Lebesgue's dominated convergence theorem yields

$$q_v^{b\varepsilon} \rightharpoonup q_v^b \text{ strongly in } L^p(0,t_1;V) \text{ for any } p > 1. \tag{4.103}$$

In particular, we will use the result with $p = \frac{5}{2}$ for passing to limit in the q_v-equation.

Now, for the q_v-equation $(4.33)_1$, the treatment will be different because of the penalization term as we will see below. Let us first rewrite as follows the weak formulation of the penalized equation $(4.33)_1$ in view of (4.28). For all $q_v^b \in \mathcal{K}(U)$, we consider $q_v^{b\varepsilon} = q_v^b - (q_v^b - q_{vs}^\varepsilon)^+ = \min(q_v^b, q_{vs}^\varepsilon) \leqslant q_{vs}^\varepsilon$. We then write the first equation (q_v−equation) of (4.36) with q_v^b replaced by $q_v^{b\varepsilon} - q_v^\varepsilon$, and we find

$$\begin{aligned}\langle \partial_t q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon\rangle + a_{q_v}(q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon) + b(\mathbf{u}, q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon) - l_{q_v}(q_v^{b\varepsilon} - q_v^\varepsilon) \\ + \frac{1}{\varepsilon_1}\langle ((q_v^\varepsilon - q_{vs}^\varepsilon)^+)^{3/2}, q_v^{b\varepsilon} - q_v^\varepsilon\rangle = (f_{q_v}(U^\varepsilon) + F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), q_v^{b\varepsilon} - q_v^\varepsilon).\end{aligned} \tag{4.104}$$

Regarding (4.104), we first observe that

$$\begin{aligned}&\langle ((q_v^\varepsilon - q_{vs}^\varepsilon)^+)^{3/2}, q_v^{b\varepsilon} - q_v^\varepsilon\rangle \\ &= \underbrace{\langle ((q_v^\varepsilon - q_{vs}^\varepsilon)^+)^{3/2}, q_v^{b\varepsilon} - q_{vs}^\varepsilon\rangle}_{\leqslant 0 \text{ (because } q_v^{b\varepsilon} \leqslant q_{vs}^\varepsilon)} + \underbrace{\langle ((q_v^\varepsilon - q_{vs}^\varepsilon)^+)^{3/2}, (q_{vs}^\varepsilon - q_v^\varepsilon)\rangle}_{\leqslant 0 \text{ (by definition of the positive function)}} \leqslant 0.\end{aligned} \tag{4.105}$$

Then, after integrating in time on $(0, t_1)$ and using (4.105), we rewrite (4.104) as follows:

$$\begin{aligned}\int_0^{t_1} \langle \partial_t q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon\rangle dt + \int_0^{t_1} a_{q_v}(q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon) dt + \int_0^{t_1} b(\mathbf{u}, q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon) dt \\ - \int_0^{t_1} l_{q_v}(q_v^{b\varepsilon} - q_v^\varepsilon) dt \geqslant \int_0^{t_1} (f_{q_v}(U^\varepsilon) + F\mathcal{H}_{\varepsilon_2}(q_v - q_{vs}), q_v^{b\varepsilon} - q_v^\varepsilon) dt.\end{aligned} \tag{4.106}$$

In what follows we will justify the passage to the limit in (4.106), term by term. First, we observe that

$$\int_0^{t_1} \langle \partial_t q_v^\varepsilon, -q_v^\varepsilon\rangle dt = -\frac{1}{2}\int_0^{t_1} \frac{d}{dt}|q_v^\varepsilon|_{L^2}^2 = -\frac{1}{2}|q_v^\varepsilon(t_1)|_{L^2}^2 + \frac{1}{2}|q_{v0}|_{L^2}^2, \tag{4.107}$$

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_0^{t_1} \langle \partial_t q_v^\varepsilon, -q_v^\varepsilon\rangle dt &= -\liminf_{\varepsilon\to 0} \frac{1}{2}|q_v^\varepsilon(t_1)|_{L^2}^2 + \frac{1}{2}|q_{v0}|_{L^2}^2 \\ &\leqslant -\frac{1}{2}|q_v(t_1)|_{L^2}^2 + \frac{1}{2}|q_{v0}|_{L^2}^2 \\ &= -\int_0^{t_1} \langle \partial_t q_v, q_v\rangle dt\end{aligned} \tag{4.108}$$

In addition, by (iii), Lemma 4.6 and Remark 4.7, we have

$$\langle \partial_t q_v^\varepsilon, q_v^{b\varepsilon}\rangle \to \langle \partial_t q_v, q_v^b\rangle, \text{ as } \varepsilon \to 0. \tag{4.109}$$

We then obtain

$$\limsup_{\varepsilon\to 0} \int_0^{t_1} \langle \partial_t q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon\rangle dt \leqslant \int_0^{t_1} \langle \partial_t q_v, q_v^b - q_v\rangle dt. \tag{4.110}$$

The convergence of the a_{q_v}−term in (4.106) is straightforward. Indeed, on the one hand, using Lemma 4.6 and the expression of a as in (4.7), we have

$$\int_0^{t_1} a_{q_v}(q_v^\varepsilon, q_v^{b\varepsilon}) dt \to \int_0^{t_1} a_{q_v}(q_v, q_v^b) dt. \tag{4.111}$$

On the other hand, we can now pass to the lower limit in the remaining part of the a_{q_v}−term and we obtain

$$\liminf_{\varepsilon\to 0}\int_0^{t_1} a_{q_v}(q_v^\varepsilon, q_v^\varepsilon)dt \geqslant \int_0^{t_1} a_{q_v}(q_v, q_v)dt. \tag{4.112}$$

Then, using the expressions of the linear and trilinear forms l and b, as in (4.10), (4.8), the convergence results in (i) and Lemma 4.6, we deduce, as $\varepsilon \to 0$ and for all $q_v^b \in \mathcal{K}$, that

$$\int_0^{t_1} b(\mathbf{u}, q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon)dt \to \int_0^{t_1} b(\mathbf{u}, q_v, q_v^b - q_v)dt, \tag{4.113}$$

$$\int_0^{t_1} l_{q_v}(q_v^{b\varepsilon} - q_v^\varepsilon)dt \to \int_0^{t_1} l_{q_v}(q_v^b - q_v)dt. \tag{4.114}$$

In fact, the convergence result (4.114) is obvious. However, for (4.113), we used the following estimates:

$$\begin{aligned}
&\left|\int_0^{t_1}[b(\mathbf{u}, q_v^\varepsilon, q_v^{b\varepsilon} - q_v^\varepsilon)dt - b(\mathbf{u}, q_v, q_v^b - q_v)]dt\right| \\
&= \left(\left|\int_0^{t_1}[b(\mathbf{u}, q_v^\varepsilon - q_v, q_v^{b\varepsilon} - q_v^\varepsilon)dt + b(\mathbf{u}, q_v, q_v^{b\varepsilon} - q_v^b) - b(\mathbf{u}, q_v, q_v^\varepsilon - q_v)]dt\right|\right) \\
&\leqslant |\mathbf{u}|_{L^\infty((0,t_1)\times\mathcal{M})}\left[|q_v^\varepsilon - q_v|_{L^2(0,t_1;V)} + |q_v^{b\varepsilon} - q_v^b|_{L^2(0,t_1;V)}\right].
\end{aligned}\tag{4.115}$$

Now for the RHS of (4.106), we use (4.101) together with (v) and Lemma 4.5 and we obtain

$$(f_{q_v}(U^\varepsilon)+F\mathcal{H}_{\varepsilon_2}(q_v-q_{vs}), q_v^{b\varepsilon}-q_v^\varepsilon) \to (f_{q_v}(U)+Fh_{q_v}, q_v^b-q_v^\varepsilon), \text{ as } \varepsilon \to 0,\ \forall\, q_v^b \in \mathcal{K}. \tag{4.116}$$

We are left to check that h_{q_v} belongs to $\mathcal{H}(q_v - q_{vs})$, i.e., to prove (4.31). For $\varepsilon_2 \in (0, 1]$, we define the real function

$$K_{\varepsilon_2}(r) = \begin{cases} 0 & for\ r \leqslant 0, \\ r^2/2\varepsilon_2 & for\ r \in (0, \varepsilon_2], \\ r - \varepsilon_2/2 & for\ r > \varepsilon_2. \end{cases} \tag{4.117}$$

It is straightforward to check that $K'_{\varepsilon_2} = \mathcal{H}_{\varepsilon_2}$, and for $\forall r_1,\ r_2 \in \mathbb{R}$

$$|\mathcal{H}_{\varepsilon_2}(r_1) - \mathcal{H}_{\varepsilon_2}(r_2)| \leqslant \frac{1}{\varepsilon_2}|r_1 - r_2|, \tag{4.118}$$

$$|K_{\varepsilon_2}(r_1) - K_{\varepsilon_2}(r_2)| \leqslant |r_1 - r_2|. \tag{4.119}$$

Moreover,

$$|K_{\varepsilon_2}(r) - r| \leqslant \frac{\varepsilon_2}{2},\ \forall r \geqslant 0. \tag{4.120}$$

Becuase $\mathcal{H}_{\varepsilon_2}(q_v^\varepsilon - q_{vs}^\varepsilon)$ is the Gâteaux derivative at the point $q_v^\varepsilon - q_{vs}^\varepsilon$ of the convex function

$$\int_0^{t_1}(K_{\varepsilon_2}(\cdot), 1)dt : L^2(0, t_1; V) \to \mathbb{R}.$$

For every ε, we have

$$\int_0^{t_1}(K_{\varepsilon_2}(q_v^{b\varepsilon}-q_{vs}^{\varepsilon}),1)dt-\int_0^{t_1}(K_{\varepsilon_2}(q_v^{\varepsilon}-q_{vs}^{\varepsilon}),1)dt \geqslant \int_0^{t_1}\langle\mathcal{H}_{\varepsilon_2}(q_v^{\varepsilon}-q_{vs}^{\varepsilon}),q_v^{b\varepsilon}-q_v^{\varepsilon}\rangle dt \quad (4.121)$$

for each $q_v^b \in L^2(0,t_1;V)$. By (v), (vii) and Lemma 4.6, we see that, as $\varepsilon \to 0^+$,

$$\int_0^{t_1}\langle\mathcal{H}_{\varepsilon_2}(q_v^{\varepsilon}-q_{vs}^{\varepsilon}),q_v^{b\varepsilon}-q_v^{\varepsilon}\rangle dt \to \int_0^{t_1}\langle h_{q_v},q_v^b-q_v\rangle dt \quad (4.122)$$

for $\forall q_v^b \in L^2(0,t_1;V)$.

Moreover, owing to (4.118) and (4.120), we observe that

$$\begin{aligned}&\Big|\int_0^{t_1}(K_{\varepsilon_2}(q_v^{\varepsilon}-q_{vs}^{\varepsilon}),1)dt-\int_0^{t_1}([q_v-q_{vs}]^+,1)dt\Big|\\ &\leqslant \int_0^{t_1}(|(K_{\varepsilon_2}(q_v^{\varepsilon}-q_{vs}^{\varepsilon})-K_{\varepsilon_2}(q_v-q_{vs})|,1)dt+\int_0^{t_1}(|K_{\varepsilon_2}(q_v-q_{vs})-[q_v-q_{vs}]^+|,1)dt\\ &\leqslant \mu(\mathcal{M})^{1/2}t_1^{1/2}(|q_v^{\varepsilon}-q_v|_{L^2(0,t_1;L^2)}+|q_{vs}^{\varepsilon}-q_{vs}|_{L^2(0,t_1;L^2)})+\frac{\varepsilon_2}{2}\mu(\mathcal{M})t_1, \end{aligned} \quad (4.123)$$

where $\mu(\mathcal{M})$ is the volume of $\mathcal{M}$. Therefore

$$K_{\varepsilon_2}(q_v^{\varepsilon}-q_{vs}^{\varepsilon}),1)dt \to \int_0^{t_1}([q_v-q_{vs}]^+,1)dt. \quad (4.124)$$

From the calculation above, it is also clear that

$$K_{\varepsilon_2}(q_v^{b\varepsilon}-q_{vs}^{\varepsilon}),1)dt \to \int_0^{t_1}([q_v^b-q_{vs}]^+,1)dt, \quad (4.125)$$

for $\forall q_v^b \in L^2(0,t_1;V)$.

Consequently, we can pass to the limit in (4.121) and conclude that

$$\int_0^{t_1}([q_v^b-q_{vs}]^+,1)dt-\int_0^{t_1}([q_v-q_{vs}]^+,1)dt \geqslant \int_0^{t_1}\langle h_{q_v},q_v^b-q_v\rangle dt \quad (4.126)$$

for $\forall q_v^b \in L^2(0,t_1;V)$, which implies (4.31) as desired.

Finally, we obtain the existence of the solution of (4.28) for which we state the following theorem.

Theorem 4.8. *Let $U_0 \in \mathbb{V}$, $t_1>0$ be given and assume that $\mathbf{u} \in L^\infty((0,t_1)\times\mathcal{M})$ is given. Then, the system (4.25)-(4.28) associated with the initial and boundary conditions (4.34) and (4.35), respectively, has a solution U such that*

$$U \in L^\infty(0,t_1;\mathbb{H}) \cap L^2(0,t_1;\mathbb{V}), \quad (4.127)$$

Furthermore, we have

$$\bar{U} \in L^2(0,t_1;H^2), \quad \partial_t\bar{U} \in L^2(0,t_1;L^2), \quad \partial_t q_v \in L^{5/3}(0,t_1;V^*). \quad (4.128)$$

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